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Ready, Set, Learn

Michael G. Wright

John Carroll University, mwright18@jcu.edu

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READY, SET[®], LEARN!

An Essay Submitted to the
Office of Graduate Studies
College of Arts & Sciences of
John Carroll University
in Partial Fulfillment of the Requirements
for the Degree of
Master of Arts

By
Michael G. Wright
2017

The creative project of Michael G. Wright is hereby accepted:

Advisor – Dr. Barbara K. D'Ambrosia

Date

I certify that this is the original document

Author – Michael G. Wright

Date

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Introduction

Making mathematics exciting for the general population can be an extremely difficult task. For a portion of the population, just seeing a proof followed by the theorem is all the excitement that they need; such is not the case for most millennials. As a high school math teacher, I am not contractually obligated to make the mathematics that I teach engaging and applicable for my students but I have found that my students are much more successful when I teach in this manner. In my short time teaching, I have spent countless hours not only trying to create material that is more appealing, but also making myself more engaging to students who are being conditioned by technology to have shorter and shorter attention spans. The driving factor behind the sequence of lessons in this essay was to use a game, SET®, to make the mathematics that I am required to teach both captivating and relevant for all of my students.

With the implementation of Ohio's new Learning Standards, courses like Algebra I and Geometry now include more Statistics and Probability than before. Also, the existence of so many Statistics and Probability standards is suggestive for high schools around the state of Ohio to offer a Statistics and Probability course to students who have completed Algebra II. This is exactly what my high school decided to do about three years ago. Statistics and Probability is a course in which juniors and seniors get a feel for the mathematics that they are most likely to see if they go into any sort of business, medical, science or research career. Understanding the graph of a parabolic function is required by the state of Ohio, but being able to manipulate a spreadsheet and perform statistical analysis has a much more real and practical application for the majority of students. In my school, students who will be pursuing math and science careers are encouraged to take Pre-Calculus, and any other student is encouraged to take Statistics and Probability during their senior year (and some students take both).

The students in my Statistics and Probability course are a diverse group of learners so it is extremely important for me to try to create a level playing field. A game like SET® does just that because it is fun and engaging for all of my students. The game itself could be used and played in any classroom even at the elementary level, but for my high school students more advanced mathematics is necessary. In the past, the foundation for my probability examples has been a standard deck of cards, but in this essay I develop a unit of lessons using a deck of SET® cards. The biggest reason for this change is that some students come into my class with a strong knowledge of a standard deck of cards. This knowledge may stem from games that their families play or personal interest in

cards; but other students do not even know how many cards are in a standard deck. The beauty of using a SET[®] deck of cards is that the chance of a student knowing anything about these cards before the beginning of the course is very low. For some students, the difference between a standard deck of cards and a SET[®] deck is inconsequential, but to me it is a way to have more of my students begin with the same amount of prior knowledge; and thus 'level the playing field'.

The first lessons in the unit are dedicated to explaining how to play SET[®], as well as uncovering many of the wonderful mathematical properties of the deck. The main section of lessons focuses on combinatorics and how this study can be used to count various things within the game of SET[®]. Although a lot of mathematics is done with the SET[®] cards in this project, I only scratch the surface. Within this seemingly simple game lie connections to advanced combinatorics, modular arithmetic, affine geometry, vectors and linear algebra. Some of those topics could spark the interest of my young scholars once they have been introduced to SET[®].

I have designed this sequence of lessons to open mathematical doors that my students did not know existed. This simple game engages students in so many ways without their realizing that they are doing math! However, once understood, the mathematics in this project will help students consider many things they did not previously see. After all, this is why I became a math teacher: to challenge students to recognize and understand the math that surrounds them in their everyday lives.

Unit at a Glance

This essay contains a unit that was designed for implementation in a high school Statistics and Probability course, but could be used in any course that discusses basic combinatorics. In particular, Ohio's End of Course Assessment for Geometry ([3]) includes Probability and Counting, and this sequence of lessons would work towards that requirement.

For each lesson in the unit, I have created a number of documents, which may include: a set of guided notes to be completed by students during class; entrance and/or exit slips, which I explain below; in-class review problems; and homework assignments. I have also created a reference sheet that contains the sample space of a SET[®] deck, a cumulative study guide that will allow students to review topics from the lessons, and a summative exam. I have designed the materials with my personal pedagogy in mind, but the resources could easily be adapted to fit any teacher's style.

The pacing guide serves as an overview of the topics and the order in which they are covered. Each lesson is meant for a single 50-minute class period and has a detailed lesson plan that contains learning objectives, materials needed, assessment techniques and the content standards that align with the lesson. Every lesson plan also explains expectations for the educator and the students, topics and strategies for students who may struggle, and an extension section for students who excel. Some lessons also include additional background information for teachers.

The shaded portions of text in the teacher documents indicate answers to the questions, definitions, and solutions to the problems. Some of the documents in the lessons contain colored images of SET[®] cards. I have added a small letter to each image of a SET[®] card indicating its color, so that color printing is not necessary. Teachers who decide to print the documents in color should keep the small letters on the cards for students who struggle to distinguish the colors. The creators of SET[®] suggest another method of differentiating the colors which they describe in [5] (p. 22).

Educators have differing opinions about student note-taking. The rationale for my guided notes system is student participation. When I take the time to make guided notes, 100% of my students are actively engaged with note-taking, as opposed to less when I use a more open-ended note taking system. Guided notes do not only increase student participation, they save time. Counting and combinatorics questions are typically word problems that students have to read

carefully. When students do not have to spend time copying down the text of a problem, they are able to spend more time processing the question. Due to the intricacies of the questions, guided notes also help insure that students have complete and correct notes that they are able to reference at all times, especially when preparing for an assessment.

I have designed the exit and entrance slips to be given at the beginning or end of a class for about five minutes. "Exit slip" has become an assessment 'buzz word', but their use conforms to best practices. For many of the lessons that I teach, I either have a pre-printed exit slip or have blank half sheets of paper and a few problems in mind that could become an exit slip instantaneously. Although I do not make exit slips high stakes, my students know to take them seriously because they paint a picture for me about the students' understanding of a topic. I collect entrance and exit slips as formative assessment and grade them as I would a homework assignment.

Flexibility is key for all instructional materials, and I kept that in mind when I designed the homework assignments for these lessons. I typically give my students 5-10 minutes at the end of a class period to start their homework and ask questions while they work. I then collect the assignment the following day, after I give the students a chance to ask more questions. There are also times when I need to slow down material, so I will give the students more class time to complete assignments, sometimes with a partner or group depending on the situation.

One of my favorite types of formative assessment involves individual whiteboards. Students respond well to working on the whiteboards because they break up the monotonous routines of notes and assignments. I have great success with whiteboard reviews mainly due to the fact that when students make a mistake on a whiteboard it is so quick and painless to correct. When I implement the in-class review problems, I display them on my SMART board and the students can work on them individually. While students are working I circle the room offering real time feedback on their work. I designed these review problems with personal whiteboards in mind, but I often have students who would prefer to do the work on notebook paper to be able to take the examples with them. With this in mind, an educator can present the problems in any way that they see fit, with the ultimate goal being practice for the students.

I typically give study guides to students 2 or 3 days prior to a summative exam. We then spend the day before the exam going over any questions that the students have after working on their study guide, followed by class time to finish

the study guide. Study guides are not worth any points in my class because the real assessment of a student's knowledge happens during the exam. I collect study guides for a reference point on a student's exam grade, and then pass them back after the exam. The summative exam in this unit will prove challenging for the majority of students, while fairly assessing their knowledge of the content in this unit.

One last note about the word *set*. Throughout this unit, a special group of three cards will be referred to as a *SET* and the game itself will be referred to as SET®. In order to describe cards succinctly in the teacher notes, I have given each feature a code, so that four symbols represent each card uniquely. The chart below shows the symbols used in this coding system, and Figures 1 and 2 show examples.

Feature	Codes
Number	1, 2, 3
Color	R (red), G (green), or P (purple)
Shading	E (empty/no shading), M (medium/striped), F (full/solid)
Shape	S (squiggle), D (diamond), or O (oval)

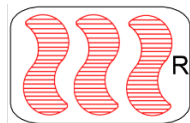


Figure 1. Card 3RMS.



Figure 2. Card 1PFD.

Ohio's Learning Standards for Mathematics

Ohio's Learning Standards in Mathematics has a Statistics and Probability section at the high school level. The counting techniques in the unit in this essay prepare students for the ninth standard under "Conditional Probability and the Rules of Probability":

S.CP.9 Use permutations and combinations to compute probabilities of compound events and solve problems.

The sequence of lessons in this unit also addresses many of the Standards of Mathematical Practices in Ohio's Learning Standards. The practices are as follows:

MP.1 Make sense of problems and persevere in solving them.

MP.2 Reason abstractly and quantitatively.

MP.3 Construct viable arguments and critique the reasoning of others.

MP.4 Model with mathematics.

MP.5 Use appropriate tools strategically.

MP.6 Attend to precision.

MP.7 Look for and make use of structure.

MP.8 Look for and express regularity in repeated reasoning.

Pacing Guide

The pacing guide that follows is based on 50-minute class periods. All of the materials listed can be found in the appendices.

Lesson 1: Introduction to SET®	Lesson 2: Fundamental Theorem of SET®
<u>Procedures</u> <ul style="list-style-type: none"> ➤ Teacher/students discuss guided notes about the game of SET® ➤ Students play SET® in small groups <u>Materials</u> <ul style="list-style-type: none"> ➤ Notes ➤ SET® decks 	<u>Procedures</u> <ul style="list-style-type: none"> ➤ Teacher/students discuss guided notes about the Fundamental Theorem of SET® ➤ Students begin working on their homework assignment in class <u>Materials</u> <ul style="list-style-type: none"> ➤ Notes, Assignment
Lesson 3: Fundamental Counting Principle	Lesson 4: The Factorial Function
<u>Procedures</u> <ul style="list-style-type: none"> ➤ Teacher/students discuss guided notes about the Fundamental Counting Principle and how to apply it ➤ Students will complete an exit slip <u>Materials</u> <ul style="list-style-type: none"> ➤ Notes, Exit Slip 	<u>Procedures</u> <ul style="list-style-type: none"> ➤ Students complete an entrance slip about simplifying fractions ➤ Teacher/students discuss guided notes about the factorial function ➤ Students begin working on their homework assignment in class <u>Materials</u> <ul style="list-style-type: none"> ➤ Entrance Slip, Notes, Assignment
Lesson 5: Permutations and Combinations	Lesson 6: Permutations and Combinations, continued
<u>Procedures</u> <ul style="list-style-type: none"> ➤ Teacher/students discuss guided notes about Permutations and Combinations ➤ Students complete an exit slip <u>Materials</u> <ul style="list-style-type: none"> ➤ Notes, Exit Slip 	<u>Procedures</u> <ul style="list-style-type: none"> ➤ Students complete whiteboard review questions about permutations and combinations ➤ Students begin working on their homework assignment in class <u>Materials</u> <ul style="list-style-type: none"> ➤ In-Class Review Problems, Assignment

Lesson 7: SET® Counting Challenge	Lesson 8: Review
<u>Procedures</u> <ul style="list-style-type: none"> ➤ Students work through counting different types of <i>SETs</i> in the deck ➤ Students begin working on their homework assignment in class <u>Materials</u> <ul style="list-style-type: none"> ➤ Activity, Assignment 	<u>Procedures</u> <ul style="list-style-type: none"> ➤ Students work towards completion of the study guide ➤ Students are given a chance to ask any questions <u>Materials</u> <ul style="list-style-type: none"> ➤ Cumulative Study Guide
Exam	
<u>Procedures</u> <ul style="list-style-type: none"> ➤ Students work individually to complete the summative exam <u>Materials</u> <ul style="list-style-type: none"> ➤ Summative exam 	

Lesson Plans

Lesson 1: Introduction to SET®	
<u>Learning Objectives</u> <ul style="list-style-type: none">➤ Students will be able to identify SETs of cards from an array➤ Students will be able to play a game of SET®	<u>The instructor will...</u> <ul style="list-style-type: none">➤ Begin class with a discussion about the game of SET®➤ Distribute blank guided notes and display the guided notes using a projector➤ Carefully explain how the examples in the notes do or do not constitute a SET➤ Challenge students to find all of the SETs in the final example, choosing students to share their responses➤ Group students for group play of the game➤ End class with a discussion about how mathematics is involved in this game
<u>Materials</u> <ul style="list-style-type: none">➤ Notes➤ SET® decks for each group	
<u>Assessment</u> <p>Student-teacher interaction during the guided notes and group play.</p>	<u>The students will...</u> <ul style="list-style-type: none">➤ Participate in the guided notes, examples and classroom discussions about the content➤ Play the game of SET®➤ Describe ways that mathematics is involved in the game of SET®
<u>Content Standards</u> <ul style="list-style-type: none">➤ MP.7➤ MP.8	
<u>Traps and Pitfalls</u> <p>Students tend to see SETs with fewer attribute differences much more quickly than SETs with more attribute differences, so the instructor should explain all of the types of SETs carefully with multiple examples.</p>	
<u>Extensions</u> <p>Teachers can challenge student groups during the game-playing phase by asking questions like:</p> <ul style="list-style-type: none">➤ How do you know that a collection of cards is a SET?➤ How many cards were left at the end of your game?➤ Must there be cards left over? Why or why not? <p>For students who excel, the teacher could introduce them to a variant of the game called the ‘End Game’: at the beginning of the game, the players hide one card and when the game is over, the player who is able to deduce which card was hidden wins.</p>	

Lesson 2: Fundamental Theorem of SET®	
<u>Learning Objectives</u> <ul style="list-style-type: none">➤ Students will be able to state and apply the Fundamental Theorem of SET®➤ Students will be able to form a plane of nine cards when given three cards that do not form a SET	<u>The instructor will...</u> <ul style="list-style-type: none">➤ Begin class with a daily SET® puzzle from the SET® Enterprises website (puzzles.setgame.com/set/puzzle_frame.htm)➤ Distribute blank guided notes and display the guided notes using a projector➤ Explain why the card that completes a SET is unique, and that this result is the Fundamental Theorem of SET®➤ Challenge students to infer how to complete a plane of SET® cards➤ Give students an opportunity to begin their homework at the end of class (time permitting)
<u>Materials</u> <ul style="list-style-type: none">➤ Notes➤ Assignment	
<u>Assessment</u> <ul style="list-style-type: none">➤ Student-teacher interaction during the guided notes➤ Graded homework assignment	
<u>Content Standards</u> <ul style="list-style-type: none">➤ MP.1➤ MP.2➤ MP.7➤ MP.8	<u>The students will...</u> <ul style="list-style-type: none">➤ Participate in the guided notes, examples and classroom discussions about the content➤ Brainstorm how to make a plane after they are given three cards that do not form a SET➤ Have an opportunity to begin their homework at the end of class
<u>Traps and Pitfalls</u> <p>Students may struggle to see all 12 SETs in a plane; if a SMART board is available, students could manipulate the cards to show all of the SETs, or the SETs could be listed out. Or the instructor could give students a SET® deck to manipulate at their desk.</p>	
<u>Extensions</u> <p>Some students may excel at visualizing cards that are missing; for those students, this would be another good time for the instructor to bring up the End Game variation described above. Another way for the teacher to challenge the students is by having them lay out all of the red cards and then count the number of SETs that exist in this arrangement. This arrangement of 27 red cards forms another special SET® structure, called a <i>hyperplane</i>. Planes and hyperplanes relate directly to affine geometry, but one interesting property that students can actually test is closure. If any two cards are chosen in a plane or hyperplane, the card that completes the SET is also in the plane or hyperplane.</p>	

Lesson 2 Teacher Notes

Readers who are unfamiliar with the rules of SET[®] should read through the materials for Lesson 1 before delving into these teacher notes. There is a strong connection between the game of SET[®] and the subject of Geometry. SET[®] is not related to the typical Euclidean geometry where the number of points, lines and planes is infinite, but it is connected to another version called finite geometry. The SET[®] Enterprises website refers to the plane in the guided notes as a “magic square”. The phrase “magic square” more commonly refers to a square array of distinct integers where each row, column and diagonal sum to the same number, though. Thus, mathematicians prefer the term “plane” for the special configuration of SET[®] cards, because of the game’s link to finite affine geometry.

Any geometry, whether Euclidean, spherical, hyperbolic, etc., is built up from a typically small number of axioms that are assumed to be true. Then theorems that further describe the geometry are proved from those original axioms. Finite affine geometry focuses on the study of parallelism and does not include metrics such as distance and angle, nor does it adhere to a concept of “straightness.” In Lesson 2, we define the affine plane $A(2,3)$, a structure that has 9 points and 12 lines, with three points in each line. This plane satisfies the axioms below:

Axiom 1: There are at least three non-collinear points.

Axiom 2: Every line contains at least two points.

Axiom 3: Two points determine a unique line.

Axiom 4: For any line m and any point P not on m , there is exactly one line containing P and not containing any point on m . Because this line is in the same plane as m , we say it is parallel to m .

In order to relate the game of SET[®] to Geometry, we will think of the cards as points and SETs as lines. Axiom 1 translates directly to the game because we can find three cards that are not in the same SET. The second Axiom states, in SET[®] terms, that every SET contains at least two cards, which should also be clear in terms of the game. Axiom 3 is equivalent to the Fundamental Theorem of SET[®]: Given two cards, there is exactly one card that completes the SET. The last axiom states that given a SET and a card not in that SET, there is exactly one SET containing the given card that is parallel to the given SET. This embodies Euclid’s famous “Parallel Postulate.” In Euclidean Geometry, nonintersecting lines are not necessarily parallel because they may not lie in the

same plane. In the same way, nonintersecting *SETs* are not necessarily parallel, as we shall see when we define the concept of “parallel” *SETs*.

We start with a description of the cyclic ordering of the cards in a *SET*. Picture a *SET* with the three cards glued around a cylinder. For example, imagine the cards in one of the *SETs* in Figure 3 glued to the curved section of a cylinder and focus on just the color of the cards. Since the cards wrap around the cylinder, after we read the third card, we read the first card again to start a new cycle. As we rotate the cylinder, we read the colors of the first *SET* as “..., red, green, purple, red, green” For two *SETs* to have the same cyclic ordering for one attribute, they must have the same left-to-right ordering of that attribute on the cylinder. The color attribute for both *SETs* in Figure 3 has cyclic ordering “..., purple, red, green, purple, red,” In contrast, the cyclic ordering for color in the *SET* from Figure 4 is “..., purple, green, red, purple, green, ... ,” which is different.

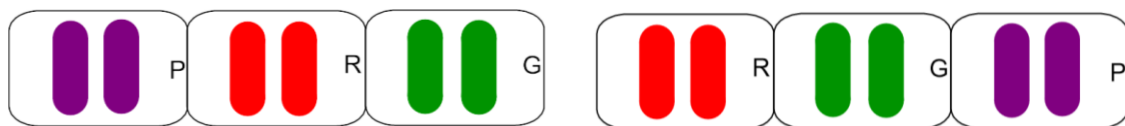


Figure 3. *SETs* with the same cyclic ordering for color.

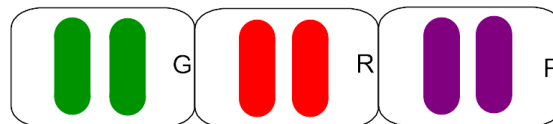


Figure 4. *SET* with different cyclic ordering for color than the *SETs* in Figure 3.

In order to understand Axiom 4, we must have a definition of “parallel” for *SETs*. We define two nonintersecting *SETs* to be parallel provided that:

- If a feature is the same on all cards in one *SET*, then it’s the same on all cards in the other.
- If a feature is different on all cards in one *SET*, then it’s different on all cards in the other.
- The cards in each *SET* can be arranged so that the cyclic ordering of each feature that is all different is the same for that feature in both *SETs*.

In Figure 5 the number is all the same in the first *SET* and in the second as well. Since color is all different in both *SETs*, we must examine the cyclic ordering of the colors. Both *SETs* have a cyclic ordering for color that is “..., red, green, purple, red, green,” Similarly for shading, we observe that the cyclic

ordering for both *SETs* is "..., solid, striped, empty, solid, striped," For the shapes on the cards, the cyclic ordering for both *SETs* is "..., ovals, squiggles, diamonds, ovals, squiggles," Thus, the two *SETs* in Figure 5 are parallel. In Figure 6, the *SETs* have all the same shape, but number, color and shading are all different. The cyclic ordering for number in both *SETs* in Figure 6 is "..., three, two, one, three, two," The cyclic orderings for color and shading are "..., green, red, purple, green, red," and "..., solid, striped, empty, solid, striped, ...", respectively. Therefore, the *SETs* in Figure 6 are parallel as well.

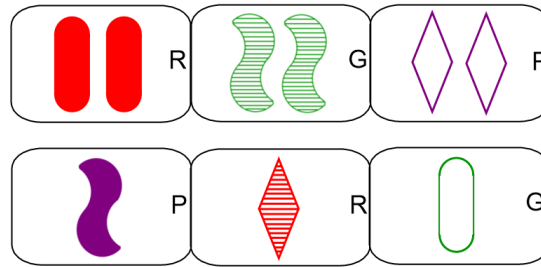


Figure 5. Parallel *SETs*.

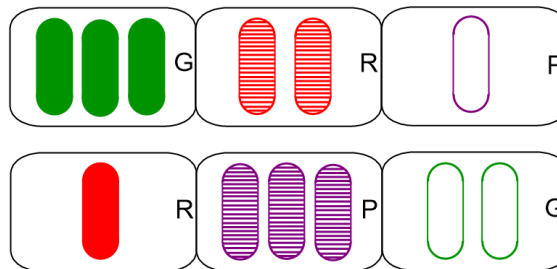


Figure 6. Parallel *SETs*.

To show that two *SETs* are not parallel, we must only determine that some aspect of the definition fails. In the case of the *SETs* in Figure 7, the *SETs* do not have the same cyclic ordering with respect to their color. As the *SETs* are arranged currently, they do have equivalent cyclic orderings for shading and shape. If we rearrange one of the *SETs* so that the cyclic order for color is the same in both *SETs*, then the cyclic orderings for shading and shape are no longer equivalent. Therefore, the *SETs* are not parallel. In Figure 8 the first *SET* has all the same number, while the second *SET* has all different numbers, and therefore the *SETs* are not parallel.

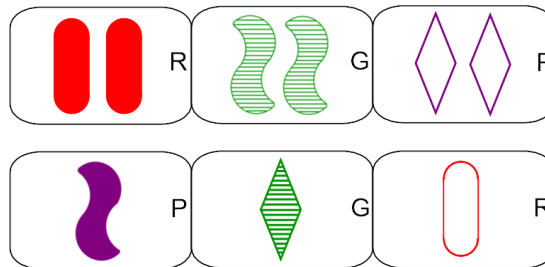


Figure 7. *SETs* that are not parallel.

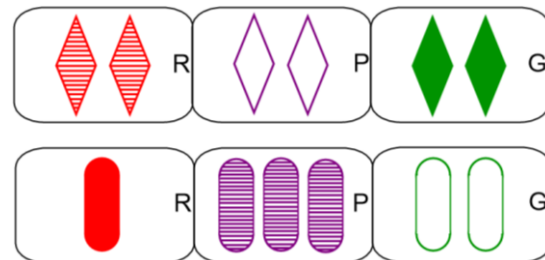


Figure 8. *SETs* that are not parallel.

Now that we understand what it means for two *SETs* to be parallel, we can explore Axiom 4 in the SET® deck. That is, given a *SET* and a card not in that *SET*, there is exactly one *SET* that can be created with the given card that is parallel to the given *SET*. This axiom holds in general because of the cyclic ordering principle of parallel *SETs*, but we demonstrate the process of constructing parallel *SETs* with an example.

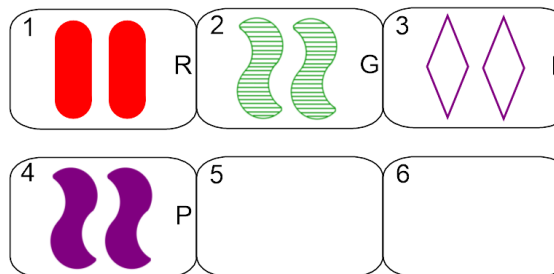


Figure 9. Four cards, three of which form a *SET*.

We begin with the four cards in Figure 9, such that cards 1, 2, and 3 form a *SET*. Our goal is to make a *SET* that is parallel to the given *SET*. We first determine card 5. Since all of the cards in the given *SET* have the same number, the cards in a parallel *SET* must all have the same number. Thus, card 5 has two shapes on it. The given *SET* has all different colors, so the parallel *SET* must also have all different colors. The cyclic ordering of the colors in the two *SETs* must match. The cyclic ordering for color in the given *SET* is “purple, red, green,” so card 5 must be red. The shading of the cards in the given *SET* is all different

with cyclic ordering “solid, striped, empty.” This implies that card 5 must be striped. Finally, we determine that the shape of card 5 must be diamond since the two *SETs* must have the same cyclic ordering for shape. Therefore, card 5 is 2RMD, and thus card 6 is 2GEO by the Fundamental Theorem of SET®. Because card 5 (and hence card 6) was uniquely determined, the *SET* parallel to the given *SET* through card 4 is unique.

Consider the three cards in Figure 10, which appear in the guided notes for Lesson 2. This group of 3 cards that are not in the same *SET* embody Axiom 1 and any three such cards could be placed in positions 1, 2 and 4. Cards 3 and 7 are determined by Axiom 3, The Fundamental Theorem of SET®, forming *SETs* in the first row and first column. Next, we place the unique card that forms a *SET* with cards 3 and 7 in position 5. When explaining this process to students, continue to reference the Fundamental Theorem of SET®. We then repeat this process for card 6, which forms a *SET* with the cards in the middle row. We can apply the same principle to cards 8 and 9 which complete *SETs* in columns 2 and 3, respectively.

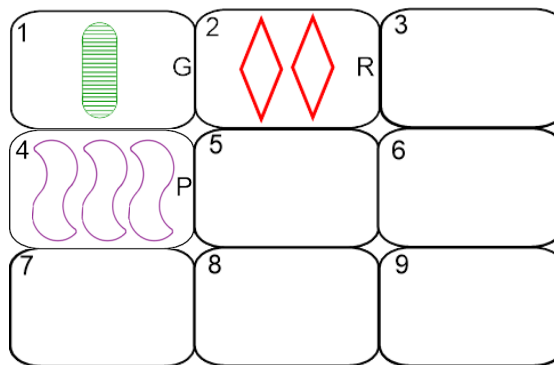


Figure 10. Three cards not in the same *SET*.

Many of the *SETs* in the completed plane turn out to be parallel. In fact all three rows will be parallel, as will all three of the columns. If we do some wrapping, all 3 “positive slope” diagonals will be parallel, as will all 3 “negative slope” diagonals. (See Figure 16)

For example, we examine why the first two rows are always parallel. Cards 1 and 4 must differ in at least one attribute, because no two SET® cards are identical. If we assume that this attribute is color, then cards 1 and 2 either have the same color or different colors. We will also arbitrarily assume that card 1 is red and that card 4 is green.

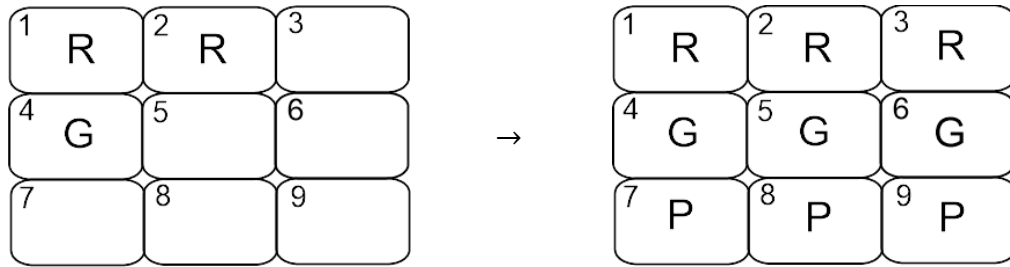


Figure 11. Cards 1 and 2 are the same color and card 4 is a different color.

Figure 11 shows the case in which cards 1 and 2 have the same color. By completing the plane we can observe that the color attribute for the *SET* in the first row is all the same, as it is for the *SET* in the second row. Thus, both *SET*s satisfy the parallel requirement for the color attribute.

Now assume again that card 1 is red and card 4 is green, but that card 2 is not red. Either card 2 is the same color as card 4, or it is not. These scenarios are illustrated below in Figures 12 and 13.

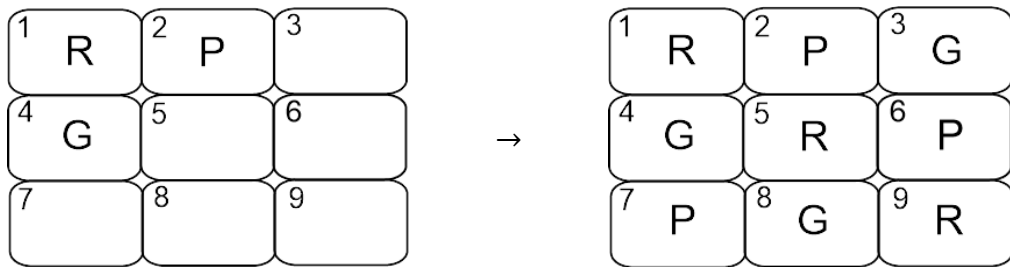


Figure 12. Cards 1 and 2 are different colors and card 4 is a third color.

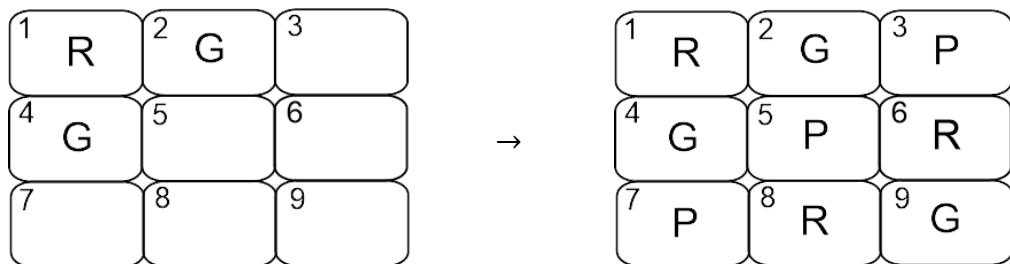


Figure 13. Cards 1 and 2 are different colors and card 4 is the same color as card 2.

In Figures 12 and 13, the color attribute is all different in both rows 1 and 2, but the cyclic ordering for color is the same in both rows.

The final case that we must consider is when cards 1 and 4 are the same in some attribute. Again, we will arbitrarily focus on color. Within this case are two situations: when cards 1 and 2 agree and when they differ. Figures 14 and 15 show the resulting planes.

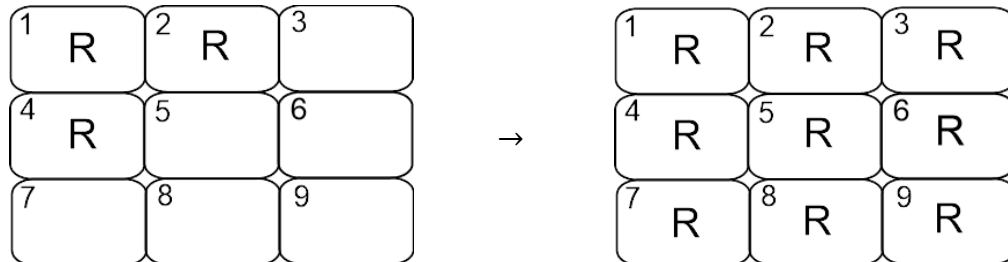


Figure 14. Cards 1, 2 and 4 are the same color.

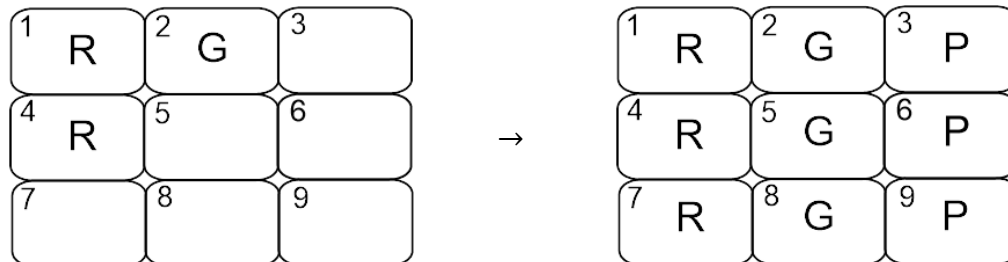


Figure 15. Cards 1 and 4 are the same color, and card 2 is a different color.

In Figures 14 and 15 we again see that the color attribute is all the same in both rows 1 and 2, or that the cyclic ordering for color is the same in both rows.

The only part of the definition of parallel *SETs* that we did not consider is the fact that the first two rows are nonintersecting. It is tedious, but not difficult, to use the Fundamental Theorem of SET® to show that each successive card in the construction of a plane is different than the cards that have already been placed.

We can replace the color attribute in the above examples by any of the other attributes. Therefore, given three cards that do not form a *SET*, a plane can be completed so that the first two rows must be parallel. We can also observe from the cases above that every pair of rows is parallel, as is every pair of columns.

The fact that the rows (respectively, columns) are all parallel accounts for 6 of the 12 *SETs* in a plane. In order to see the other collections of parallel *SETs* in a plane, we must be able to visualize *SETs* diagonally. The image in Figure 16 is a representation of a SET® plane in the geometrical sense. The nine points

represent cards and the four groups of parallel lines (*SETs*) are indicated by the same color and line type (dotted or solid).

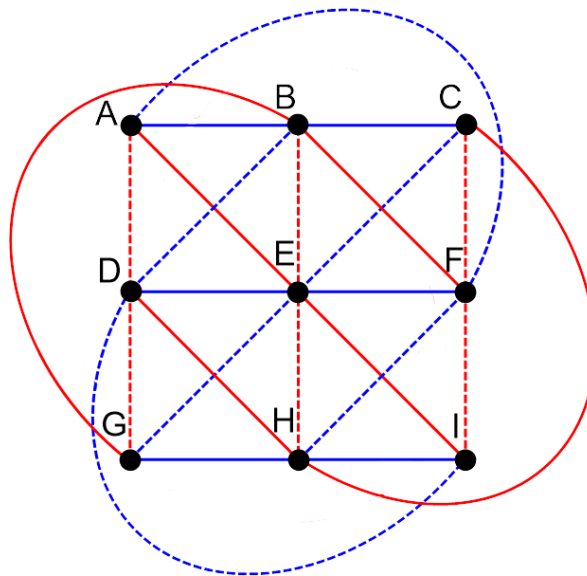


Figure 16. The affine plane with three points per line.

In every result from each case above, we can observe that all of the diagonal *SETs* of the same type must be parallel. The fact that all of the lines in the plane in Figure 16 have to exist (Axiom 3) is also known as “closure”: Every point is collinear with each other point in the plane. In *SET*[®] terms, this means that every card in a plane is contained in exactly one *SET* with every other card in the plane.

The majority of high school students will not be interested in the explanation of the affine plane. I have found that there are typically one or two students who will be fascinated with this concept of a plane with relationship to *SET*[®] cards. They will be the ones who simply can’t help asking “why?”

Lesson 3: Fundamental Counting Principle	
<u>Learning Objectives</u> <ul style="list-style-type: none"> ➤ Students will be able to state and apply the Fundamental Counting Principle to count the number of outcomes in given situations ➤ Students will be able to relate this principle to real world situations 	<u>The instructor will...</u> <ul style="list-style-type: none"> ➤ Begin class with a SET® puzzle (puzzles.setgame.com/set/puzzle_frame.htm) ➤ Collect and answer questions about Lesson 2: Assignment ➤ Distribute blank guided notes and display the guided notes using a projector ➤ Explain the Fundamental Counting Principle ➤ Relate this principle to the game of SET® and calculate the number of cards in a deck ➤ Distribute and collect the exit slip
<u>Materials</u> <ul style="list-style-type: none"> ➤ Notes ➤ Exit Slip 	
<u>Assessment</u> <ul style="list-style-type: none"> ➤ Student-teacher interaction during the guided notes ➤ Graded exit slip 	<u>The students will...</u> <ul style="list-style-type: none"> ➤ Participate in the guided notes, examples and classroom discussions about the content ➤ Attempt the examples on their own when asked, and be ready to share their work verbally as well as on the board ➤ Complete the exit slip to the best of their ability
<u>Content Standards</u> S.CP.9	
<u>Traps and Pitfalls</u> Students often are trapped by the Fundamental Counting Principle because the calculation aspect is so rudimentary. When they realize the pattern, many students will stop reading the rest of the problem and just multiply all of the numbers that are in the problem. Be sure that students understand what and why they are multiplying; visuals, including tree diagrams, are helpful for students who struggle with the rationale for multiplying.	
<u>Extensions</u> If students finish the exit slip too quickly, the teacher can challenge them to change something else about the deck to make more cards. Let the students be creative and even sketch their new cards on the blank side of the exit slip.	

Lesson 4: The Factorial Function	
<u>Learning Objectives</u> ➤ Students will be able to simplify and rewrite expressions using factorials	<u>The instructor will...</u> ➤ Begin class with an entrance slip ➤ Have students trade entrance slips and grade each other's work ➤ Distribute the SET® Deck Reference Sheet and explain the instructions on the sheet ➤ Distribute blank guided notes and display the guided notes using a projector ➤ Define the factorial function and have students expand and simplify factorial expressions ➤ Have students work out examples individually, then go to the board to show their work to the rest of the class ➤ End class with time to begin the assignment that is due the next class period
<u>Materials</u> ➤ Entrance Slip ➤ Notes ➤ Assignment ➤ SET® Reference Sheet	
<u>Assessment</u> ➤ Student-teacher interaction during the guided notes ➤ Graded Entrance Slip ➤ Graded homework assignment	<u>The students will...</u> ➤ Complete and grade the entrance slip ➤ Participate in the guided notes, examples and classroom discussions about the content ➤ Attempt the examples on their own when asked, and be ready to share their work verbally as well as on the board ➤ Begin homework assignment in class and ask questions when necessary
<u>Content Standards</u> S.CP.9	
<u>Traps and Pitfalls</u> Since the factorial notation is new to most, if not all, students, the teacher should solve problems thoroughly. With practice, students will be able to skip steps and 'cancel' terms quickly; in this introductory lesson they should be rigorous and explain why cancelling is possible.	
<u>Extensions</u> Have additional, more challenging problems available for students who solve the problems in the notes quickly. For example: $\frac{6!}{(6-2)!}$, $\frac{12!}{5!(12-5)!}$, or $\frac{7!3!}{(2!)!} \div \frac{8!5!(9-9)!}{(4!)^2}$	

Lesson 5: Permutations and Combinations	
<u>Learning Objectives</u> <ul style="list-style-type: none"> ➤ Students will be able to identify and distinguish permutations or combinations in context ➤ Students will be able to calculate the number of permutations or combinations for a given real world example 	<u>The instructor will...</u> <ul style="list-style-type: none"> ➤ Begin class with a few simple factorial examples that students will complete in their notes ➤ Collect and answer questions about Lesson 4: Assignment ➤ Distribute blank guided notes and display the guided notes using a projector ➤ Define <i>permutation</i> and derive the formula for the number of permutations ➤ Use the formula to calculate number of permutations in various contexts ➤ Define <i>combination</i> and derive the formula for the number of combinations ➤ Use the formula to calculate number of combinations in various contexts ➤ Distribute and collect the exit slip
<u>Materials</u> <ul style="list-style-type: none"> ➤ Notes ➤ Exit Slip 	
<u>Assessment</u> <ul style="list-style-type: none"> ➤ Student-teacher interaction during the guided notes ➤ Graded exit slip 	<u>The students will...</u> <ul style="list-style-type: none"> ➤ Participate in the guided notes, examples and classroom discussions about the content ➤ Attempt the examples on their own when asked, and be ready to share their work verbally as well as on the board ➤ Complete the exit slip to the best of their ability
<u>Content Standards</u> S.CP.9	
<u>Traps and Pitfalls</u> The biggest pitfall for students during this lesson is recognizing the difference between permutations and combinations. Refer to various concrete examples that are relevant to your students. Point out that the number of permutations is greater than the number of combinations when the students are considering the same parameters.	
<u>Extensions</u> Have students write a real world example that involves a permutation and/or combination. They should then solve their problem(s).	

Lesson 5 Teacher Notes

During this lesson it is crucial for students to understand how the formulas for the numbers of permutations and combinations are derived in order to avoid rote memorization of procedures. If students are able to understand the intuitive ways to calculate the number of permutations or combinations, then the formulas follow directly. In order to illustrate this intuitive description of the calculations, I use concrete examples. This explanation is far from a formal mathematical argument for a formula or rule but to high school students, abstract notation and explanations only serve to complicate relatively simple concepts. Depending on the students in a class, the teacher can present a more formal argument for these formulas.

The formula for the number of permutations is derived from the Fundamental Counting Principle. The first example in the guided notes has the students list six cards from an array of six SET® cards. We can solve this problem by considering how many cards there are left to choose from after each card has been placed in the arrangement and relating this expression to factorial notation. The next example is about listing only two cards from the same original array. Using the same concept, we can arrive at the answer of 30 ways by multiplying $6 \cdot 5$ instead of computing $6!$. The key to developing the formula is to explain to students how the expression $6 \cdot 5$ can be written in factorial notation $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{6!}{4!}$. At this point, ask the students to explore the significance of the 4 in the denominator of the fraction (4 is the number of cards not chosen).

Note that $(6 - 2)! = 4!$. We now have the expression $\frac{6!}{(6 - 2)!}$. In general, the

number of permutations of r items chosen from a set of n items is

$${}_nP_r = \frac{n!}{(n-r)!}.$$

At the end of the permutation section in the notes, there are several examples in which students calculate the number of distinguishable permutations of the letters from a given word. When I explain this process to my students, I begin with the words “TRY” and “WOW” in order to show the effect of letters that repeat. To guide students through the development of a formula for the number of distinguishable permutations of the letters in a word, I use the example “CLASSES”. First, I have students rewrite the word with subscripts on the Ss: “CLAS₁S₂ES₃”. When students apply the Fundamental Counting Principle to find

the number of unique ways to list the letters in the word now, the words that they are counting are unique. In order to show the effect that these letters have when permuting, I write one permutation of all the letters on the board with “A”, “L” and “C” fixed (e.g. “CLAS₃S₁ES₂”). Note that “CLAS₁S₂ES₃” and “CLAS₃S₁ES₂” are not unique permutations without the subscripts. Next, I permute the three Ss without changing the location of the other letters. For each distinguishable arrangement of the letters C, L, A, S, S, E, S, there are $3! = 6$ ways to permute the indistinguishable Ss. Thus, when we count the number of permutations of 6 letters, we have overcounted the number of distinguishable words by a factor of $3! = 6$. Hence, the number of distinguishable permutations of the letters in the word “CLASSES” is $\frac{6!}{3!} = 120$.

In general, the number of distinguishable permutations of a set that has some indistinguishable elements is calculated by the formula:

$$\frac{(\text{total number of elements})!}{(\text{freq. of } 1^{\text{st}} \text{ element})!(\text{freq. of } 2^{\text{nd}} \text{ element})! \cdots (\text{freq. of last element})!}.$$

The formula for the number of combinations follows directly from the formula for the number of permutations. The only difference is that the formula for the number of combinations has an additional $r!$ term in the denominator. When we divide by $r!$, we eliminate the overcounting that occurs when different permutations are the same combination. In the notes, students write all of the permutations of the three cards in a *SET* and notice that each permutation is the same *SET*. Thus all six permutations are the same combination. The second example in this portion of the notes leads students to the realization that when we use ${}_nP_r$ to count a number of permutations, we are counting each combination $r!$ times. The number of combinations of r elements taken from a set of n elements is thus the number of permutations, divided by $r!$:

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}.$$

In my experience, there are students who do not memorize the formulas because they prefer to develop the computation for each question intuitively. Either way, students are more likely to comprehend and remember the formula if they have seen the rationale. Without going through the process of deriving formulas, students would simply be substituting numbers into a formula and

following a procedure to compute a result as opposed to developing a real understanding of how and why that result is correct.

Lesson 6: Permutations and Combinations, continued	
<u>Learning Objectives</u> <ul style="list-style-type: none">➤ Students will be able to identify and distinguish situations involving permutations or combinations➤ Students will be able to calculate the number of permutations or combinations for a given real world example	<u>The instructor will...</u> <ul style="list-style-type: none">➤ Begin class with two computational examples to check students' understanding: one permutation and one combination.➤ Distribute personal whiteboards, dry erase markers and erasers to the students➤ Display problems on the board and circulate the room to check answers and constantly give feedback to students on their work➤ If certain problems are particularly challenging, go over those problems as a class, or have a student reproduce their correct work on the board for the class to see➤ End class with time to begin the assignment that is due the next class period
<u>Materials</u> <ul style="list-style-type: none">➤ In-class review problems➤ Assignment	
<u>Assessment</u> <ul style="list-style-type: none">➤ Student-teacher interaction during the in-class review problems➤ Graded homework assignment	<u>The students will...</u> <ul style="list-style-type: none">➤ Participate in the whiteboard review by completing problems on their personal whiteboard, and then displaying their work to the instructor➤ Use local resources including the instructor appropriately➤ Begin the homework assignment in class and ask questions when necessary
<u>Content Standards</u> S.CP.9	
<u>Traps and Pitfalls</u> <p>The instructor should again emphasize the difference between permutations and combinations. Use as many examples as possible for students who are struggling with this concept. Also stress the differences in the formulas and how the results are computed.</p>	
<u>Extensions</u> <p>Have students write a real world example that involves a permutation and/or combination. They should then solve their problem(s). If students did this during the previous lesson, have students exchange their problems with other students, who will solve the problems, and then the author of the problems can check the solution.</p>	

Lesson 7: SET® Counting Challenge	
<u>Learning Objectives</u> <ul style="list-style-type: none"> ➤ Students will be able to calculate the number of <i>SETs</i> in the SET® deck ➤ Students will be able to calculate the number of different types of <i>SETs</i> in the full SET® deck 	<u>The instructor will...</u> <ul style="list-style-type: none"> ➤ Begin this lesson by asking students to list all of the permutations of their favorite three SET® cards. ➤ Collect and answer questions about Lesson 6: Assignment ➤ Distribute blank activity papers and display the teacher activity page using a projector ➤ Have students complete the activity document using one of the strategies listed in the guided notes ➤ End class with time to begin the assignment that is due the next class period
<u>Materials</u> <ul style="list-style-type: none"> ➤ Activity Document ➤ Assignment 	
<u>Assessment</u> <ul style="list-style-type: none"> ➤ Student-teacher interaction during the activity ➤ Graded homework assignment 	<u>The students will...</u> <ul style="list-style-type: none"> ➤ Participate in the activity, asking questions as they progress ➤ Collaborate with peers to count the different <i>SETs</i> within the game ➤ Begin the homework assignment in class and ask questions when necessary
<u>Content Standards</u> S.CP.9	
<u>Traps and Pitfalls</u> The teacher can adapt this counting challenge for students at various levels. However the instructor decides to present this material, the most confusing parts will be recognizing and avoiding overcounting, and having to use the Fundamental Counting Principle with numbers of combinations for the later examples. The instructor should use more examples for these topics when necessary.	
<u>Extensions</u> There are any number of counting questions to extend this lesson that are extremely complicated, but one straightforward extension would be to have the students count the following quantities just for the red cards: <ul style="list-style-type: none"> ➤ How many red cards are there? ➤ How many red <i>SETs</i> are there? ➤ Among the red cards, how many <i>SETs</i> have one, two or three attributes different? 	

Lesson 7 Teacher Notes

This type of lesson is my favorite to teach because it takes the content standards that educators are required to follow and pushes them to their limits. This lesson also gives the instructor more creative freedom. After reading through the lesson plan, materials, and teacher notes, the instructor should be able to combine that information with the knowledge of their students' abilities and interests in order to decide how or whether to present this lesson.

The most structured way to teach this lesson is for the teacher to present the activity document as they would guided notes. This strategy allows students to see and hear a content area expert explain the steps necessary to calculate the different examples. If a teacher employs this method of presentation for the lesson, they can gauge their students' understanding along the way. At any point during the lesson, the instructor can transition the students to working through the examples on their own or with groups. In contrast to this method of instruction, the instructor could instead arrange students into small groups and allow them to attempt the calculations with one another. Regardless of the format, the lesson offers students an example of slightly more complicated combinatorics and another application of the counting principles learned thus far in the lessons.

The first example in the activity document has students calculate the number of *SETs* in the full *SET*® deck. This procedure involves the Fundamental Counting Principle and the Fundamental Theorem of *SET*®. The most likely part of this calculation that may act as a pitfall to students is the reason for dividing the number of ordered *SETs* by $3!$. This follows directly from the derivation of the formula for the number of combinations. If a student is struggling with the rationale, remind them of the number of ways to order the cards in one *SET* and that counting all of those listings as distinct *SETs* would be overcounting by a factor of $3! = 6$.

Another error students might make is in counting the number of combinations of three cards from the whole deck: ${}_{81}C_3 = 85,320$. The instructor should remind students that most of these collections of three cards are not *SETs*, though and thus this number is much too large.

An alternate way to calculate the number of *SETs* in the deck is to recall that each *SET* is determined by two cards; and any 2 cards determine a *SET*. So first we calculate all of the ways to choose two cards from the deck in any order: ${}_{81}C_2 = 3,240$. We are once again overcounting, though. We must divide by the

number of ways to choose two cards that determine the same *SET*. In fact, there are ${}_3C_2 = 3$ ways to choose two cards from a *SET* of three cards. So the number of *SETs* in the full deck is $\frac{{}_{81}C_2}{{}_3C_2} = 1,080$.

Throughout this counting challenge, it is important for teachers to let their students have creative freedom as well. Every student is a unique individual who may think of solving a problem from a different angle, so the instructor should embrace that within this and all lessons.

The calculation of the number of *SETs* in which all four attributes are different should follow directly from the calculation for the number of *SETs*. The big difference is the restriction on the second card chosen, but since each attribute has two other options the calculation should not offer too many issues. For calculating the number of *SETs* with exactly three attributes different, the students must be able to visualize all four different options for the 2nd card. If students do not understand why they must multiply by 4 ($= {}_4C_1$), then it may help for the instructor to write out the other options for the second card. This calculation would be tedious, but for students who have never done calculations like this before, seeing all of the work could be exactly what they need in order to truly understand the calculations. Finding the correct answer for these calculations is important, but if students are not truly grasping the process, then they have not met the real lesson objective.

For the final two calculations, the students should be left on their own or in small groups, even if the teacher presented the rest of the activity as guided notes. If students need clarification, then the instructor can allot some time during the next lesson for reviewing the assignment and possibly the activity document for Lesson 7.

Lesson 8: Review	
<u>Learning Objectives</u> ➤ Students can count outcomes using the Fundamental Counting Principle, the factorial function, permutations and combinations	<u>The instructor will...</u> ➤ Collect and answer questions about Lesson 7: Assignment ➤ Ask students if there are questions in the study guide that need to be explained ➤ Students are allowed to work either together or individually on problems in the study guide
<u>Materials</u> ➤ Cumulative Study Guide	
<u>Assessment (Summative)</u> Problems from lessons 3-7.	<u>The students will...</u> ➤ Review Lesson 7: Assignment ➤ Have some problems from the study guide completed before class and be ready to ask questions of the instructor ➤ Work towards completion of their study guide
<u>Content Standards</u> S.CP.9	

Summative Exam	
<u>Learning Objectives</u> ➤ Students can count outcomes using the Fundamental Counting Principle, the factorial function, permutations and combinations	<u>The instructor will...</u> ➤ Administer the summative exam
<u>Materials</u> ➤ Summative exam	
<u>Assessment (Summative)</u> Problems from lessons 3-7.	<u>The students will...</u> ➤ Complete the summative exam during the allotted time
<u>Content Standards</u> S.CP.9	

Conclusion

The first time that I played the game SET[®] was while I was taking my graduate level courses at John Carroll University. The game absolutely intrigued me to the point that I bought the book *Joy of SET[®]* ([2]) so that I could further investigate the mathematics within the game. The fact that the game itself appears so simple fuels my intrigue, but there is a vast range of mathematical complexities hidden within the game. Ultimately, my goal for creating this sequence of lessons was to take the subject of combinatorics and make it more engaging, relevant, and interesting to my students. In the future, I plan to expand the unit to include lessons about probability.

Out of curiosity, I introduced SET[®] to my 5-and 7-year-old niece and nephew and they were engaged and asking math related questions even without being prompted. My hope is that this engagement and interest level would translate directly to high school students with the difference that high school students would ask higher-level mathematics questions. The game aspect is what makes this unit engaging for any student and I have yet to meet someone who does not enjoy looking for SETs faster than their opponent. In my experience, if something truly interests my students, then they will ask questions about it to explore more beyond the surface. The mathematics in the game of SET[®] uncovers itself naturally as students become more interested in the game and start to ask more in-depth questions.

Oftentimes in probability questions, we use a standard deck of cards for examples because information about a deck of cards was once perceived as common knowledge. In my experience, only a small percentage of my students come into my Statistics and Probability course with any knowledge of a standard deck of cards. Most students remember that there are 52 cards and not much else. Instead of teaching my students about a standard deck of cards, this sequence of lessons levels the playing field by using cards that no one has likely seen before. SET[®] cards embody fewer choices for each attribute than standard playing cards (e.g. 3 numbers instead of 13), so probability questions involving SET[®] are more tractable than with standard cards. Even students who have played the game of SET[®] probably have not explored the complex mathematics hidden within the game. All of my students, regardless of their prior knowledge, will need to learn about the game and the cards to be successful throughout this sequence of lessons.

Mathematics is not always the most popular subject in school, but I have found that there are usually several students in every math class who have a genuine love for math. These are the students who stare at the derivation of the quadratic formula in awe and the students who will work on any math problem they are given no matter the complexity. I did not design this sequence of lessons for these students specifically, but through the use of extensions located at the end of each lesson, an inquisitive young scholar could explore some mathematics topics that high school classes do not typically address. For the rest of the class, the fun and newness associated with the game of SET® will act as a 'hook' for the lessons. Beyond that, all students will collaborate, explore, and use the SET® cards to further expand their understanding of combinatorics. By relating math to fun and interesting topics, I am able to boost both interest and confidence levels among my students on a daily basis.

In today's educational landscape, teachers are not given much latitude in the content that they must teach. In my school, teachers are told *what* to teach, but we are given a lot of freedom to decide *how* we want to teach. The unit described in this essay is a wonderful example of how an instructor can take a topic they enjoy and marry it with content standards that are dictated by the State. I have truly enjoyed building this sequence of lessons and I cannot wait to implement it in my classes in the coming year.

Appendix A: Lesson Materials

Name: _____ Date: _____

Lesson 1: Introduction to SET®

Notes (*Student Document*)

Rules of SET®:

The object of the game is to identify a *SET* of three cards from the 12 cards laid out on the table. Each card has a variation of the following four attributes (features).

Number:

Color:

Symbol:

Shading:

Definition:

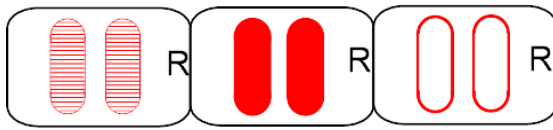
A *SET* consists of three cards in which each individual attribute is either the same on each card or different on all three cards. That is, any attribute in the *SET* of three cards is either common to all three cards or all three versions of that attribute appear in the *SET*.

Coding System:

Below is a chart that allows us to refer to each card as a code instead of drawing the card.

Feature	Codes
Number	
Color	
Shading	
Shape	

Examples of SETs:

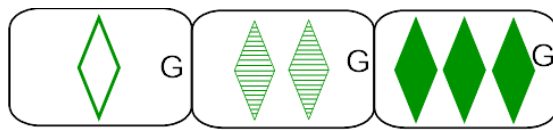


Number:

Color:

Symbol:

Shading:



Number:

Color:

Symbol:

Shading:

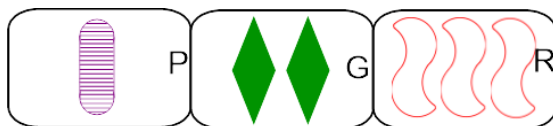


Number:

Color:

Symbol:

Shading:



Number:

Color:

Symbol:

Shading:



Number:

Color:

Symbol:

Shading:

Examples of non-SETs:



Number:

Color:

Symbol:

Shading:

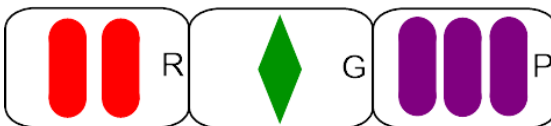


Number:

Color:

Symbol:

Shading:



Number:

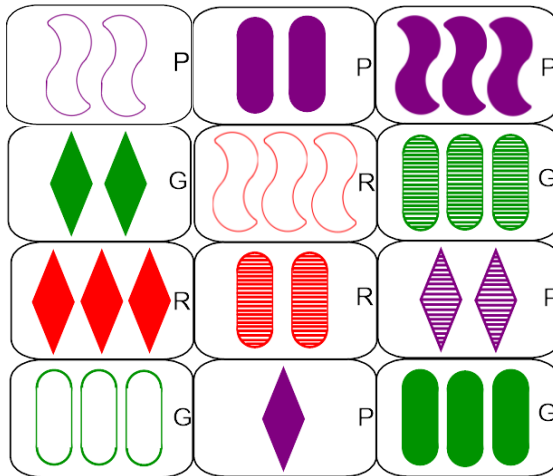
Color:

Symbol:

Shading:

SET® Puzzle:

Find and record as many of the *SETs* in the card array below as you can. The cards do not have to be in the same row or column in order to form a *SET*. Some cards will be present in more than one *SET*.



SETs found:

Reference for this document:

"Daily SET® Puzzle." SET® Enterprises. SET® Enterprises, 28 Apr. 2017. Web. 28 Apr. 2017. <puzzles.setgame.com/se/puzzle_frame.htm>.

Name: _____ Date: _____

Lesson 1: Introduction to SET®

Notes (Teacher Document)

Rules of SET®:

The object of the game is to identify a *SET* of three cards from the 12 cards laid out on the table. Each card has a variation of the following four attributes (features).

Number: Each card has one, two or three symbols

Color: The symbols on each card are a common color: red, green or purple

Symbol: The symbols on each card are a common shape: oval, squiggle or diamond

Shading: The symbols on each card have a common shading: solid (full), empty (none) or striped (medium)

Definition:

A *SET* consists of three cards in which each individual attribute is either the same on each card or different on all three cards. That is, any attribute in the *SET* of three cards is either common to all three cards or all three versions of that attribute appear in the *SET*.

Coding System:

Below is a chart that allows us to refer to each card as a code instead of drawing the card.

Feature	Codes
Number	1, 2, 3
Color	R (red), G (green), or P (purple)
Shading	E (empty/no shading), M (medium shading/striped), F (full shading/solid)
Shape	S (squiggle), D (diamond), or O (oval)

Examples of SETs:

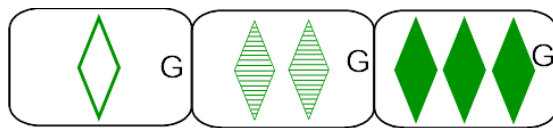


Number: all SAME

Color: all SAME

Symbol: all SAME

Shading: all DIFFERENT



Number: all DIFFERENT

Color: all SAME

Symbol: all SAME

Shading: all DIFFERENT

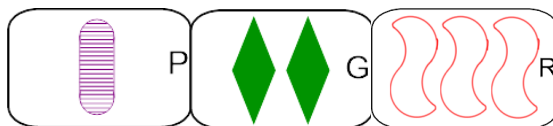


Number: all DIFFERENT

Color: all DIFFERENT

Symbol: all DIFFERENT

Shading: all SAME



Number: all DIFFERENT

Color: all DIFFERENT

Symbol: all DIFFERENT

Shading: all DIFFERENT



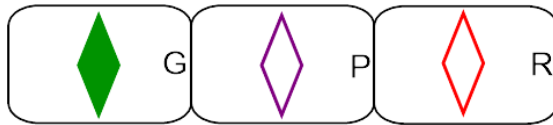
Number: all SAME

Color: all SAME

Symbol: all SAME

Shading: all DIFFERENT

Examples of non-SETs:

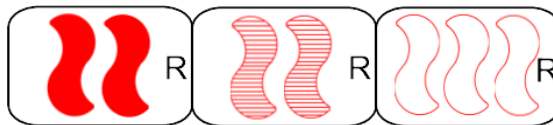


Number: all SAME

Color: all DIFFERENT

Symbol: all SAME

Shading: two empty, one solid

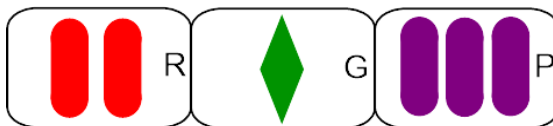


Number: two 2s, one 3

Color: all SAME

Symbol: all SAME

Shading: all DIFFERENT



Number: all DIFFERENT

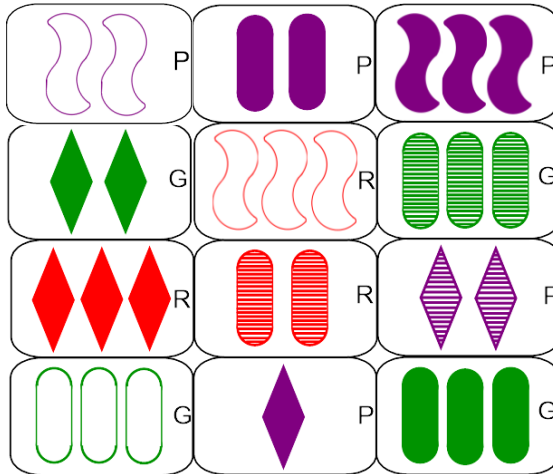
Color: all DIFFERENT

Symbol: 2 oval, 1 diamond

Shading: all SAME

SET® Puzzle:

Find and record as many of the *SETs* in the card array below as you can. The cards do not have to be in the same row or column in order to form a *SET*. Some cards will be present in more than one *SET*.



SETs found:

1PFD, 2PFO, 3PFS
3GEO, 3GMO, 3GFO
2PES, 2GFD, 2RMO
1PFD, 2GFD, 3RFD
3PFS, 3RFD, 3GFO
2PES, 2PFO, 2PMD

Reference for this document:

"Daily SET® Puzzle." SET® Enterprises. SET® Enterprises, 28 Apr. 2017. Web. 28 Apr. 2017. <puzzles.setgame.com/se/puzzle_frame.htm>.

Name: _____ Date: _____

Lesson 2: Fundamental Theorem of SET®

Notes (*Student Document*)

Complete the *SET*:

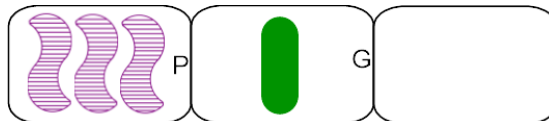
For each pair of cards below identify a third card necessary to complete a *SET*:



Did anyone else in the room find a different card to complete the *SET*?



Did anyone else in the room find a different card to complete the *SET*?



Did anyone else in the room find a different card to complete the *SET*?

Fundamental Theorem of SET®:

Apply the Fundamental Theorem of SET®:

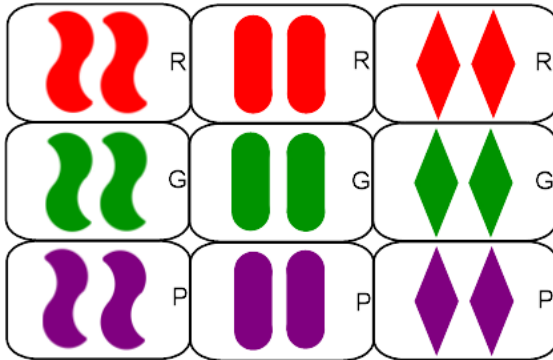
For the pair of cards below identify the third card necessary to complete a *SET*:



Identifying SETs:

List all of the SETs that you can find in the array below:

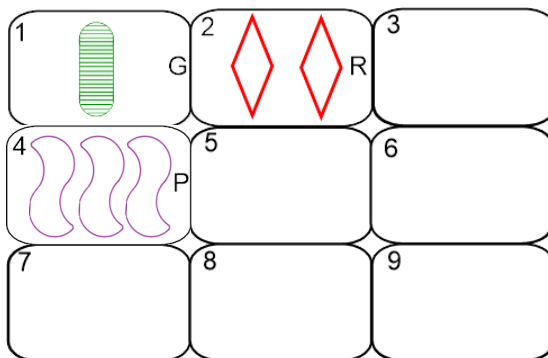
SETs found:



The SET® Plane:

The nine-card array above is called a _____. It contains the most SETs possible in an array of nine cards.

To make a plane with ____ cards and _____, we begin with three cards that _____.

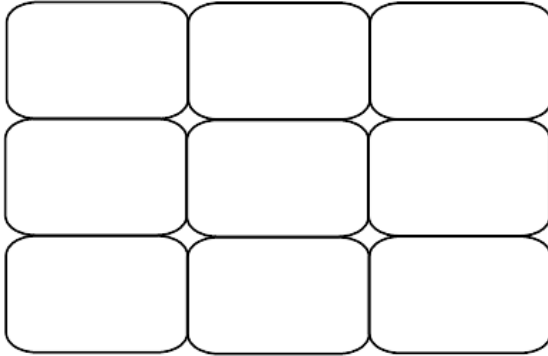


Follow the procedure below in order to complete the plane above.

- Place the three starting cards in positions 1, 2 and 4.
- Fill in card 3 to make the top row a SET.
- Fill in card 7 to make the first column form a SET.
- Fill in card 5 to form a SET with cards 3 and 7.
- Card 6 forms a SET with the middle row.

- Card 8 forms a *SET* with the middle column.
- Card 9 forms a *SET* with the last row.

Choose any three cards that do not form a *SET*, place them in the three spaces as in the example above and create a plane using the steps above.



SETs created (12 total in a plane):

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 14-17.

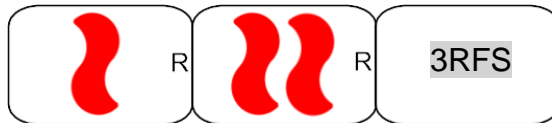
Name: _____ Date: _____

Lesson 2: Fundamental Theorem of SET®

Notes (Teacher Document)

Complete the SET:

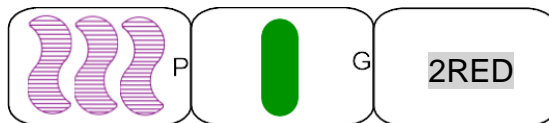
For each pair of cards below identify a third card necessary to complete a SET:



Did anyone else in the room find a different card to complete the SET? No.



Did anyone else in the room find a different card to complete the SET? No.



Did anyone else in the room find a different card to complete the SET? No.

Fundamental Theorem of SET®:

Given any pair of SET® cards, there is a unique third card that completes a SET with the given pair.

Apply the Fundamental Theorem of SET®:

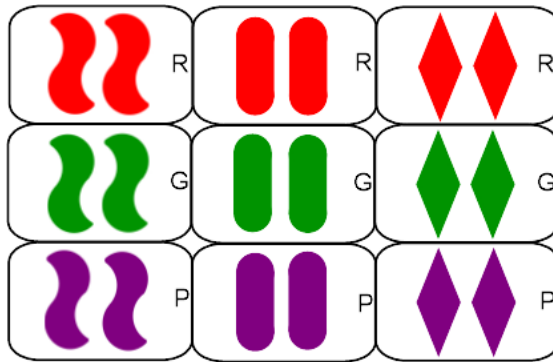
For the pair of cards below identify the third card necessary to complete a SET:



Identifying SETs:

List all of the SETs that you can find in the array below:

SETs found:

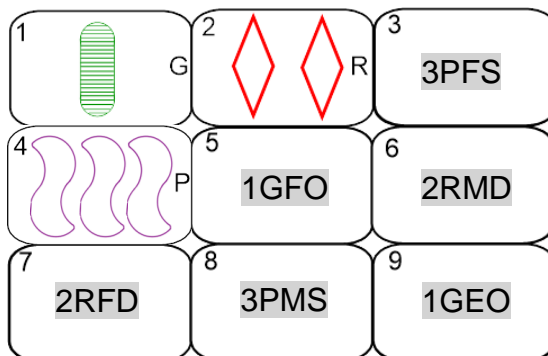


2RFS, 2RFO, 2RFD
 2GFS, 2GFO, 2GFD
 2PFS, 2PFO, 2PFD
 2RFS, 2GFS, 2PFS
 2RFO, 2GFO, 2PFO
 2RFD, 2GFD, 2PFD
 2RFD, 2GFO, 2PFS
 2RFO, 2GFS, 2PFD
 2RFS, 2GFD, 2PFO
 2RFS, 2GFO, 2PFD
 2RFD, 2GFS, 2PFO
 2RFO, 2GSD, 2PFS

The SET® Plane:

The nine-card array above is called a plane. It contains the most SETs possible in an array of nine cards.

To make a plane with 9 cards and 12 SETs, we begin with three cards that do not form a SET.

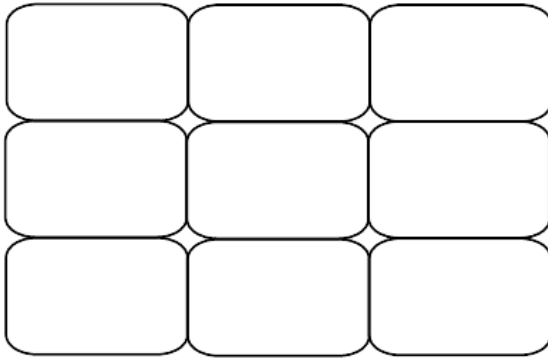


Follow the procedure below in order to complete the plane above.

- Place the three starting cards in positions 1, 2 and 4.
- Fill in card 3 to make the top row a SET.
- Fill in card 7 to make the first column form a SET.
- Fill in card 5 to form a SET with cards 3 and 7.
- Card 6 forms a SET with the middle row.

- Card 8 forms a *SET* with the middle column.
- Card 9 forms a *SET* with the last row.

Choose any three cards that do not form a *SET*, place them in the three spaces as in the example above and create a plane using the steps above.



SETs created (12 total in a plane):

Student answers will vary. Refer to Figure 16 in the teacher notes on page 18 for a comparison to finite affine geometry to help the teacher check answers.

Reference for this document:

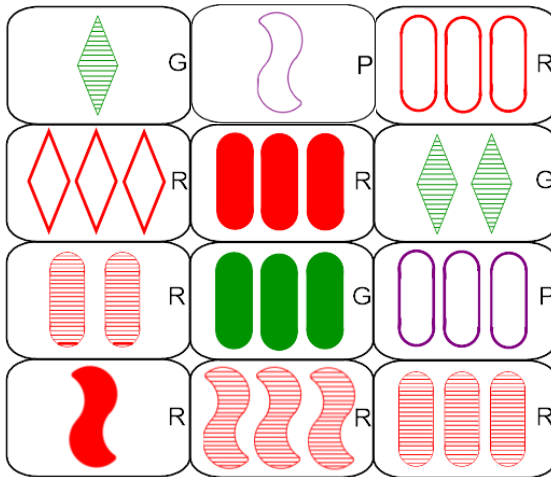
McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 14-17.

Name: _____ Date: _____

Lesson 2: Fundamental Theorem of SET®

Assignment (*Student Document*)

1. List the six *SETs* that are found in the array below:



SETs found:

2. For each pair of cards below, describe the card that will complete the *SET*:

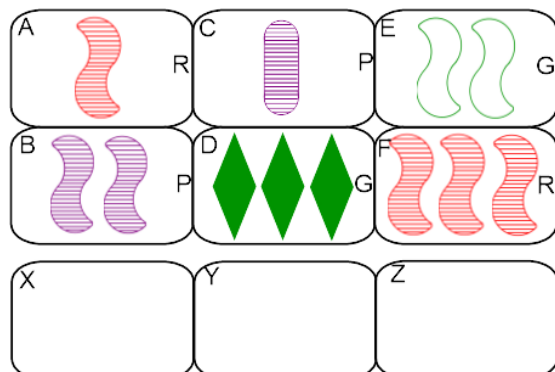
a.

b.

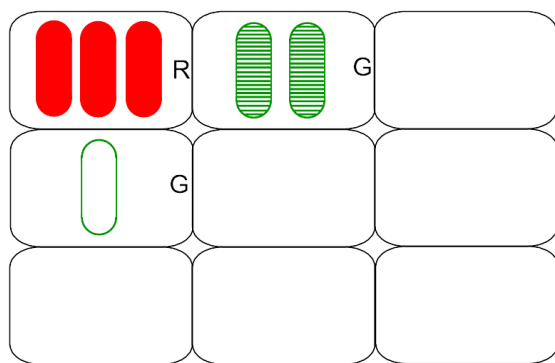
c.

d.

3. You and your friends have been playing a rousing game of SET®! The following six cards are laid out, and there are no cards remaining in the deck.



- Why is the game over?
 - Describe card X such that it makes a *SET* with cards A and B.
 - Describe card Y such that it makes a *SET* with cards C and D.
 - Describe card Z such that it makes a *SET* with cards E and F.
4. Complete the plane below:



5. How many *SET*s can be found in the plane above?

Reference for this document:

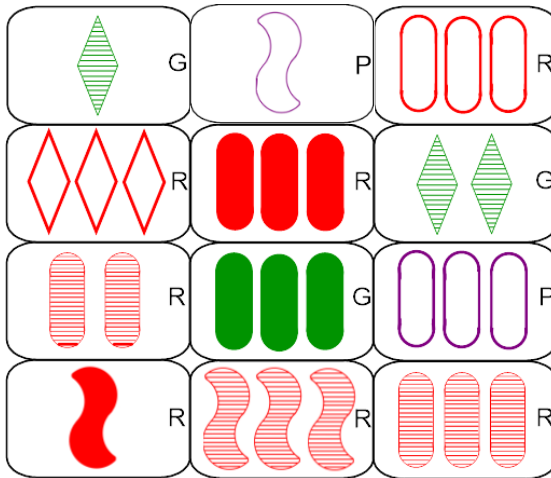
McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 14-17.

Name: _____ Date: _____

Lesson 2: Fundamental Theorem of SET®

Assignment (Teacher Document)

1. List the six *SETs* that are found in the array below:



SETs found:

3RFO, 3RMO, 3REO
1PES, 2GMD, 3RFO
3RFO, 3RED, 3RMS
3RMO, 3PEO, 3GFO
1RFS, 2RMO, 3RED
1RFS, 2GMD, 3PEO

2. For each pair of cards below, describe the card that will complete the *SET*:

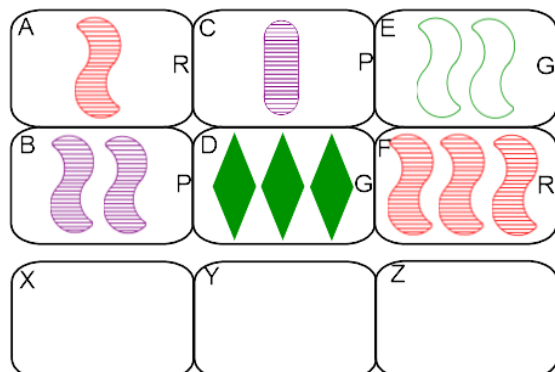
a. 2GMS

b. 2RMD

c. 1RFD

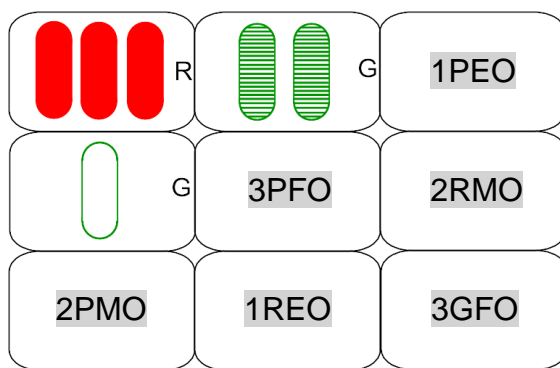
d. 1RMS

3. You and your friends have been playing a rousing game of SET®! The following six cards are laid out, and there are no cards remaining in the deck.



- Why is the game over?
There are no more *SETs* left on the board.
- Describe card X such that it makes a *SET* with cards A and B.
3GMS
- Describe card Y such that it makes a *SET* with cards C and D.
2RES
- Describe card Z such that it makes a *SET* with cards E and F.
1PFS

4. Complete the plane below:



5. How many *SETs* can be found in the plane above? 12

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 14-17.

Name: _____ Date: _____

Lesson 3: Fundamental Counting Principle

Notes (*Student Document*)

Definition:

Sample Space:

Example:

Write the sample space of sentences that can be made from filling in the blanks with the words that are shown below the lines:

My _____ is _____.
 friend brilliant
 teacher curious

Sentence sample space:

How many sentences would there be in the sample space if 'screaming' were added as a possibility for the second blank?

How many sentences would there be in the sample space if 'cousin' and 'doctor' were added as possibilities to the first blank and the 2nd blank just had the original two choices?

Fundamental Counting Principle:

Example:

Consider the following sentence:

My _____ is _____.

wealthy	cousin	always	helpful
funny	principal	sometimes	realistic
	teammate		serious

Calculate the number of sentences that would be possible for the situation above.

Example:

Elaine needs to do laundry! She has only 5 clean shirts and 4 clean pants left in her closet. How many shirt-pant combinations are available to her to wear today?

Example:

The new iPad is available in Wi-Fi or Wi-Fi 3G, and with 16GB, 32GB, or 64GB of memory. How many different iPad configurations are possible?

The Fundamental Counting Principle and SET[®]:

SET[®] cards differ in four attributes

- **Number:** 1, 2 or 3
- **Color:** green, purple or red
- **Shading:** empty, striped or solid
- **Shape:** diamond, oval or squiggle

Every card in the SET[®] deck is unique (e.g. there is only one card with 1 Red Empty Diamond) and every possible combination occurs in the deck.

1. How many cards in the SET® deck have 1 Red Diamond? Describe them below:
2. How many cards in the SET® deck have 2 Red Diamonds? Describe them below:
3. How many cards in the SET® deck have 3 Red Diamonds? Describe them below:

For problems 4-8, verify results using the Fundamental Counting Principle

4. How many cards in the SET® deck have Red Diamonds?
5. How many cards in the SET® deck have Red Squiggles?
6. How many cards in the SET® deck are Red?
7. How many cards in the SET® deck are Green with solid shading?
8. How many cards in the SET® deck are Purple with striped ovals?
9. Write the options for the attributes in the following sentences about a card from the game SET®:

I have a card with _____.

one
two
three

10. Based on the sentence in Question 9 and the Fundamental Counting Principle, how many cards are there in a complete SET® deck? (Note: all card in the SET® deck are unique.)

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 28-29.

Name: _____ Date: _____

Lesson 3: Fundamental Counting Principle

Notes (*Teacher Document*)

Definition:

Sample Space: a list of all the possible outcomes for an experiment or situation.

Example:

Write the sample space of sentences that can be made from filling in the blanks with the words that are shown below the lines:

My _____ is _____.
 friend brilliant
 teacher curious

Sentence sample space:

My friend is brilliant.
My friend is curious.
My teacher is brilliant.
My teacher is curious.

How many sentences would there be in the sample space if 'screaming' were added as a possibility for the second blank?

$$2 \cdot 3 = 6$$

How many sentences would there be in the sample space if 'cousin' and 'doctor' were added as possibilities to the first blank and the 2nd blank just had the original two choices?

$$4 \cdot 2 = 8$$

Fundamental Counting Principle:

If one event has m different outcomes, and another event has n different outcomes, then the two events together have $m \cdot n$ different outcomes.

Example:

Consider the following sentence:

My _____ is _____.

wealthy	cousin	always	helpful
funny	principal	sometimes	realistic
	teammate		serious

Calculate the number of sentences that would be possible for the situation above.

$$2 \cdot 3 \cdot 2 \cdot 3 = 36$$

Example:

Elaine needs to do laundry! She has only 5 clean shirts and 4 clean pants left in her closet. How many shirt-pant combinations are available to her to wear today?

$$5 \cdot 4 = 20 \text{ shirt-pant combinations}$$

Example:

The new iPad is available in Wi-Fi or Wi-Fi 3G, and with 16GB, 32GB, or 64GB of memory. How many different iPad configurations are possible?

$$2 \cdot 3 = 6 \text{ iPad configurations}$$

The Fundamental Counting Principle and SET[®]:

SET[®] cards differ in four attributes

- **Number:** 1, 2 or 3
- **Color:** green, purple or red
- **Shading:** empty, striped or solid
- **Shape:** diamond, oval or squiggle

Every card in the SET[®] deck is unique (e.g. there is only one card with 1 Red Empty Diamond) and every possible combination occurs in the deck.

1. How many cards in the SET[®] deck have 1 Red Diamond? Describe them below:
1RED, 1RFD, 1RMD
2. How many cards in the SET[®] deck have 2 Red Diamonds? Describe them below:
2RED, 2RFD, 2RMD
3. How many cards in the SET[®] deck have 3 Red Diamonds? Describe them below:
3RED, 3RFD, 3RMD

For problems 4-8, verify results using the Fundamental Counting Principle

4. How many cards in the SET[®] deck have Red Diamonds? $3 \cdot 3 = 9$
5. How many cards in the SET[®] deck have Red Squiggles? $3 \cdot 3 = 9$
6. How many cards in the SET[®] deck are Red? $3 \cdot 3 \cdot 3 = 27$
7. How many cards in the SET[®] deck are Green with solid shading? $3 \cdot 3 = 9$
8. How many cards in the SET[®] deck are Purple with striped ovals? 3
9. Write the options for the attributes in the following sentences about a card from the game SET[®]:

I have a card with _____.

one	purple	striped	diamonds
two	red	solid	squiggles
three	green	empty	ovals

10. Based on the sentence in Question 9 and the Fundamental Counting Principle, how many cards are there in a complete SET[®] deck? (Note: all card in the SET[®] deck are unique.)

$$3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81 \text{ cards}$$

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET[®]: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 28-29.

Name: _____ Date: _____

Lesson 3: Fundamental Theorem of SET®

Exit Slip (*Student Document*)

1. Imagine a deck of SET® cards that has one more choice for each attribute, meaning that there are cards with four shapes, and orange cards and checkerboard cards and cards with pentagons on them. Assuming that there is one of each possible card, how many cards would be in this deck?

2. Imagine a deck of SET® cards that has a fifth attribute: texture. You now have cards that feel smooth, rough and slimy. Assuming that there is one of each possible card, how many cards would be in this deck?

Name: _____ Date: _____

Lesson 3: Fundamental Theorem of SET®

Exit Slip (*Teacher Document*)

1. Imagine a deck of SET® cards that has one more choice for each attribute, meaning that there are cards with four shapes, and orange cards and checkerboard cards and cards with pentagons on them. Assuming that there is one of each possible card, how many cards would be in this deck?

$$4 \cdot 4 \cdot 4 \cdot 4 = 4^4 = 256 \text{ cards}$$

2. Imagine a deck of SET® cards that has a fifth attribute: texture. You now have cards that feel smooth, rough and slimy. Assuming that there is one of each possible card, how many cards would be in this deck?

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5 = 243 \text{ cards}$$

Name: _____ Date: _____

Lesson 4: The Factorial Function

Entrance Slip (*Student Document*)

Simplify the following expressions and turn the page over when you are finished:

1. $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

2. $\frac{5 \cdot 4}{2 \cdot 5}$

3. $\frac{8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 7}$

4. $\frac{10 \cdot 4}{20 \cdot 15} \cdot \frac{5 \cdot 2}{45}$

Name: _____ Date: _____

Lesson 4: The Factorial Function

Entrance Slip (*Teacher Document*)

Simplify the following expressions and turn the page over when you are finished:

1. $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

2. $\frac{5 \cdot 4}{2 \cdot 5} = 2$

3. $\frac{8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 7} = 8$

4. $\frac{10 \cdot 4}{20 \cdot 15} \cdot \frac{5 \cdot 2}{45} = \frac{4}{135}$

Name: _____ Date: _____

Lesson 4: The Factorial Function

Notes (*Student Document*)

Definition:

Example:

Fill in the table below:

Example	Expanded Form	Simplified

Simplify the following expressions. Use a calculator for the even numbered problems.

1. $\frac{6!}{4!}$

2. $\frac{8!}{0!}$

3. $\frac{10!}{12!}$

4. $\frac{21!}{11!}$

5. $\frac{3!4!}{6!}$

6. $(4!)!$

7. $3!2!$

8. $\frac{15!}{2!7!}$

Rewrite the following expressions using factorial notation.

9. $10 \cdot 9 \cdot 8 \cdot 7$

10. $93 \cdot 92 \cdot 91$

11. $19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14$

12. $85 \cdot 84 \cdot 83$

13. 20·19·18·9·8·7

14. 16·15·49·48

Name: _____ Date: _____

Lesson 4: The Factorial Function

Notes (Teacher Document)

Definition: The factorial of a non-negative integer n , denoted by $n!$, is the product of all the positive integers less than or equal to n . The factorial of 0 is defined to be 1; that is, $0! = 1$.

Example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Fill in the table below:

Example	Expanded Form	Simplified
$0!$	*By definition, $0! = 1$	1
$1!$	1	1
$2!$	$2 \cdot 1$	2
$3!$	$3 \cdot 2 \cdot 1$	6
$4!$	$4 \cdot 3 \cdot 2 \cdot 1$	24
$5!$	$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	120
$6!$	$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	720
$7!$	$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	5,040
$8!$	$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	40,320
$9!$	$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	362,880
$10!$	$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	3,628,800
$n!$	$n \cdot (n-1)! = n \cdot (n-1) \cdot (n-2) \cdots 1$	$n!$
$(2n)!$	$2n \cdot (2n-1) \cdot (2n-2) \cdots 1$	$(2n)!$
$30!$	$30 \cdot 29 \cdot 28 \cdots 1$	$\approx 2.62 \times 10^{32}$
$(3!)^2$	$3! \cdot 3! = (3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)$	36

Simplify the following expressions. Use a calculator for the even numbered problems.

$$1. \quad \frac{6!}{4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{6 \cdot 5}{1} = 30$$

$$2. \quad \frac{8!}{0!} = 40,320$$

$$3. \quad \frac{10!}{12!} = \frac{10 \cdot 9 \cdots 2 \cdot 1}{12 \cdot 11 \cdot 10 \cdots 1} = \frac{1}{12 \cdot 11} = \frac{1}{132}$$

$$4. \quad \frac{21!}{11!} \approx 1.28 \times 10^{12}$$

$$5. \quad \frac{3!4!}{6!} = \frac{(3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{5}$$

$$6. \quad (4!)! \approx 6.20 \times 10^{23}$$

$$7. \quad 3!2! = 12$$

$$8. \quad \frac{15!}{2!7!} = 129,729,600$$

Rewrite the following expressions using factorial notation.

$$9. \quad 10 \cdot 9 \cdot 8 \cdot 7 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdots 1}{6 \cdots 1} = \frac{10!}{6!}$$

$$10. \quad 93 \cdot 92 \cdot 91 = \frac{93!}{90!}$$

$$11. \quad 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 = \frac{19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdots 1}{13 \cdots 1} = \frac{19!}{13!}$$

$$12. \quad 85 \cdot 84 \cdot 83 = \frac{85!}{82!}$$

$$13. \quad 20 \cdot 19 \cdot 18 \cdot 9 \cdot 8 \cdot 7 = \frac{(20 \cdot 19 \cdot 18 \cdot 17 \cdots 1)(9 \cdot 8 \cdot 7 \cdot 6 \cdots 1)}{(17 \cdots 1)(6 \cdots 1)} = \frac{20!9!}{17!6!}$$

$$14. \quad 16 \cdot 15 \cdot 49 \cdot 48 = \frac{16!49!}{14!47!}$$

Name: _____ Date: _____

Lesson 4: The Factorial Function

Assignment (*Student Document*)

Simplify the following expressions. Show all work. You may use a calculator to estimate final answers for which the numerator or denominator is larger than 1000.

1. $3!$

2. $7!$

3. $10!$

4. $0!$

5. $4! + 5!$

6. $10! - 6!$

7. $8! \cdot 3!$

8. $(3!)!$

9. $\frac{7!}{5!}$

10. $\frac{4!}{6!}$

11. $\frac{12!}{15!}$

12. $\frac{(3!)^2}{6!}$

$$13. \quad \frac{10!}{5! \cdot 4!}$$

$$14. \quad \frac{77!}{80!}$$

$$15. \quad \frac{38! \cdot 3!}{39!}$$

$$16. \quad \frac{(6!)^2}{(4!)!}$$

Name: _____ Date: _____

Lesson 4: The Factorial Function

Assignment (Teacher Document)

Simplify the following expressions. Show all work. You may use a calculator to estimate final answers for which the numerator or denominator is larger than 1000.

1. $3! = 3 \cdot 2 \cdot 1 = 6$

2. $7! = 5,040$

3. $10! = 10 \cdots 1 = 3,628,800$

4. $0! = 1$

5. $4! + 5! = (4 \cdot 3 \cdot 2 \cdot 1) + (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 144$

6. $10! - 6! = 3,628,080$

7. $8! \cdot 3! = (8 \cdots 1)(3 \cdot 2 \cdot 1) = 241,920$

8. $(3!)! = 720$

9. $\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdots 1}{5 \cdots 1} = \frac{7 \cdot 6}{1} = 42$

10. $\frac{4!}{6!} = \frac{1}{30}$

11. $\frac{12!}{15!} = \frac{1}{2,730}$

12. $\frac{(3!)^2}{6!} = \frac{1}{20}$

$$13. \quad \frac{10!}{5! \cdot 4!} = 1260$$

$$14. \quad \frac{77!}{80!} = \frac{1}{492,960}$$

$$15. \quad \frac{38! \cdot 3!}{39!} = \frac{2}{13}$$

$$16. \quad \frac{(6!)^2}{(4!)!} \approx 8.36 \times 10^{-19}$$

Name: _____ Date: _____

Lesson 5: Permutations and Combinations

Notes (*Student Document*)

Definition:

Permutation (of a set):

Example:

Given 6 SET[®] cards, in how many ways can you arrange them in a row?

Example:

How many different ways are there to arrange two SET[®] cards in a row when they are chosen from an array of 6 cards?

${}_nP_r$ denotes the number of permutations of n elements taken r at a time ($r \leq n$)

--

Example:

In how many ways can the *Harry Potter* books be arranged on a bookshelf?

Example:

Twelve people need to be photographed. Five people will be seated in a row of chairs and the others will be standing behind the chairs. In how many ways can you arrange people on the five chairs?

Permutations with repetitions:**Example:**

List all the permutations of the letters: "TRY"

Example:

List all the permutations of the letters: "WOW"

Why are the numbers of permutations different for the two three-letter words above?

Example:

Find the number of distinguishable permutations of the letters in each word:

a.) GEOMETRY

b.) MISSISSIPPI

c.) MATHEMATICS

What is the fundamental difference between (b) and (c)?

Formula:

Definition:

Combination:

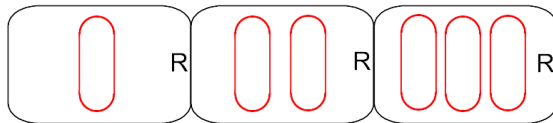
Example:

Which of the following lists below describe the same combination?

- Calculus, Abstract Algebra, Non-Euclidean Geometry
- Calculus, Differential Equations, Multi-Variable Calculus
- Linear Algebra, Non-Euclidean Geometry, Calculus
- Non-Euclidean Geometry, Calculus, Abstract Algebra
- Abstract Algebra, Calculus, Number Theory

Example:

These three cards form a *SET*. Describe all of the permutations of this *SET*.

**Example:**

How many different combinations of three SET[®] cards are possible when they are chosen from an array of 6 cards?

${}_nC_r$ denotes the number of combinations of n elements taken r at a time ($r \leq n$)

Example:

A team of 25 softball players needs to choose three players to refill the water cooler. How many different groups of three people could be chosen?

Example:

In how many different ways could Mr. Wright choose ten of his fifty students to ride in a van instead of a bus for the field trip?

Example:

If a company has twenty members of Management, in how many ways could a committee of five managers be selected?

Challenge:

Kim's Ice Cream Kastle serves 10 flavors of ice cream, 4 kinds of syrup and 6 varieties of toppings. How many different sundaes can you make if each has 2 flavors of ice cream, 2 kinds of syrup and 3 toppings?

Name: _____ Date: _____

Lesson 5: Permutations and Combinations

Notes (Teacher Document)

Definition:

Permutation (of a set): An ordered arrangement (or list) of elements of that set.

Example:

Given 6 SET® cards, in how many ways can you arrange them in a row?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720 \text{ ways}$$

Example:

How many different ways are there to arrange two SET® cards in a row when they are chosen from an array of 6 cards?

$$6 \cdot 5 = \frac{6 \cdot 5 \cdot (4 \cdot 3 \cdot 2 \cdot 1)}{(4 \cdot 3 \cdot 2 \cdot 1)} = \frac{6!}{4!} = \frac{6!}{(6-4)!} = 30 \text{ ways}$$

${}_nP_r$ denotes the number of permutations of n elements taken r at a time ($r \leq n$)

The same elements listed in different orders yield different permutations. When order matters, we're counting permutations.

$${}_nP_r = \frac{n!}{(n-r)!},$$

where n is the total number of elements to choose from and r is the number of elements being listed.

Example:

In how many ways can the *Harry Potter* books be arranged on a bookshelf?

Since there are 7 books in the series...

$${}_7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 5,040 \text{ ways}$$

Example:

Twelve people need to be photographed. Five people will be seated in a row of chairs and the others will be standing behind the chairs. In how many ways can you arrange people on the five chairs?

$${}_{12}P_5 = \frac{12!}{(12-5)!} = \frac{12!}{7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 210 \text{ ways}$$

Example:

List all the permutations of the letters: "TRY"

TRY, TYR, YTR, YRT, RTY, RYT

Example:

List all the permutations of the letters: "WOW"

WOW, WWO, OWW

Why are the numbers of permutations different for the two three-letter words above?

There are more permutations for "TRY" because the two Ws in "WOW" are indistinguishable. (If we replace the T,R,Y with W,O,W respectively, "TRY" and "YRT" both become "WOW".)

Find the number of distinguishable permutations of the letters in each word:

d.) GEOMETRY $\frac{8!}{2!} = 20,160$

e.) MISSISSIPPI $\frac{11!}{4!4!2!} = 34,650$

f.) MATHEMATICS $\frac{11!}{2!2!2!} = 4,989,600$

What is the fundamental difference between (b) and (c)?

The number of times the letters repeat.

Formula for the number of permutations with some elements indistinguishable:

$$\frac{(\text{total number of elements})!}{(\text{freq. of } 1^{\text{st}} \text{ element})!(\text{freq. of } 2^{\text{nd}} \text{ element})! \cdots (\text{freq. of last element})!}$$

Definition:

Combination: An unordered set of elements.

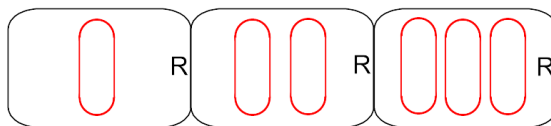
Example:

Which of the following lists below describe the same combination?

- Calculus, Abstract Algebra, Non-Euclidean Geometry
- Calculus, Differential Equations, Multi-Variable Calculus
- Linear Algebra, Non-Euclidean Geometry, Calculus
- Non-Euclidean Geometry, Calculus, Abstract Algebra
- Abstract Algebra, Calculus, Number Theory

Example:

These three cards form a *SET*. Describe all of the permutations of this *SET*.



1REO, 3REO, 2REO (given)
 1REO, 3REO, 2REO
 3REO, 1REO, 2REO

2REO, 3REO, 1REO
 2REO, 1REO, 3REO
 3REO, 2REO, 1REO

*These are all the *same combination*. *

Example:

How many different combinations of three SET® cards are possible when they are chosen from an array of 6 cards?

$${}_6C_3 = \frac{{}_6P_3}{3!} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} = 20$$

*Number of ways to list 3 things is $3 \cdot 2 \cdot 1 = 3! = 6$ ways

${}_nC_r$ denotes the number of combinations of n elements taken r at a time ($r \leq n$)

The elements of a combination can be listed in *any order*!
 Avoid counting the same combination more than once.

$${}_nC_r = \frac{n!}{r!(n-r)!},$$

where n is the total number of elements to choose from and r is the number of elements in the combination.

Example:

A team of 25 softball players needs to choose three players to refill the water cooler. How many different groups of three people could be chosen?

$${}_{25}C_3 = \frac{25!}{3!(25-3)!} = \frac{25!}{3!22!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \dots \cdot 1}{3 \cdot 2 \cdot 1 \cdot 22 \cdot 21 \cdot 20 \cdot \dots \cdot 1} = 2,300 \text{ water groups}$$

Example:

In how many different ways could Mr. Wright choose ten of his fifty students to ride in a van instead of a bus for the field trip?

$${}_{50}C_{10} = \frac{50!}{10!(50-10)!} = \frac{50!}{10!40!} = \frac{50 \cdot 49 \cdot 48 \cdot \dots \cdot 1}{10 \cdot \dots \cdot 1 \cdot 40 \cdot 39 \cdot 38 \cdot \dots \cdot 1} \approx 1.03 \times 10^{10} \text{ ways}$$

Example:

If a company has twenty members of Management, in how many ways could a committee of five managers be selected?

$${}_{20}C_5 = \frac{20!}{5!(20-5)!} = \frac{20!}{5!15!} = \frac{20 \cdot 19 \cdot 18 \cdot \dots \cdot 1}{5 \cdot \dots \cdot 1 \cdot 15 \cdot \dots \cdot 1} = 15,504 \text{ committees}$$

Challenge:

Kim's Ice Cream Kastle serves 10 flavors of ice cream, 4 kinds of syrup and 6 varieties of toppings. How many different sundaes can you make if each has 2 flavors of ice cream, 2 kinds of syrup and 3 toppings?

$$({}_{10}C_2) \cdot ({}_4C_2) \cdot ({}_6C_3) = 45 \cdot 6 \cdot 20 = 5,400 \text{ sundaes}$$

(Fundamental Counting Principle used here)

Name: _____ Date: _____

Lesson 5: Permutations and Combinations

Exit Slip (*Student Document*)

Simplify the following expressions:

1. ${}_7P_5$

2. ${}_{14}C_5$

3. A baseball team has nine players. How many different batting orders are possible, assuming that every player will be allowed to bat?
4. Eight people enter a cookie contest at the county fair, in which the top three entries advance to the state fair. In how many ways can the top three finalists be chosen as an unranked group?

Name: _____ Date: _____

Lesson 5: Permutations and Combinations

Exit Slip (*Teacher Document*)

Simplify the following expressions:

1. ${}_7P_5 = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2,520$

2. ${}_{14}C_5 = \frac{14!}{5!(14-5)!} = \frac{14!}{5!9!} = 2,002$

3. A baseball team has nine players. How many different batting orders are possible, assuming that every player will be allowed to bat?

$${}_9P_9 = 362,880 \text{ batting orders}$$

4. Eight people enter a cookie contest at the county fair, in which the top three entries advance to the state fair. In how many ways can the top three finalists be chosen as an unranked group?

$${}_8C_3 = 56 \text{ finalist groups}$$

Lesson 6: Permutations and Combinations, continued

In-Class Review Problems (*Student Document*)

Number of Permutations	Number of Combinations

For problems 1-4, compute the number of permutations and combinations, using a calculator when necessary.

1. ${}_5P_3$

2. ${}_{10}C_2$

3. ${}_{20}C_5$

4. ${}_{40}P_{35}$

5. A student council has 5 seniors, 4 juniors, 2 sophomores, and 2 freshmen as members. In how many ways can a 4-member council committee be formed that includes one member of each class?

6. How many passwords of five symbols can be made if each password must start with two digits and end three letters given that...

a. letters and digits may be repeated?

b. neither the letters nor the digits may be repeated?

7. Carla wants to take 6 different courses next year. There are 12 courses to choose from and assuming that they are all offered at different times, how many different course loads could she have?
8. A baseball team has 15 players. How many 9-player batting orders are possible?
9. A student activity club has 32 members. In how many different ways can the club select a 5-person committee?
10. A baseball team has 15 players. How many 3-player groups can be chosen to run laps around the field?
11. A student activity club has 32 members. In how many different ways can the club elect a president, a vice president, a treasurer, and a secretary?
12. A suitcase contains 6 distinct pairs of socks and 4 distinct pairs of pants. In how many ways can a traveler randomly pick 2 pairs of socks and then 3 pairs of pants?

Lesson 6: Permutations and Combinations, continued

In-Class Review Problems (*Teacher Document*)

Number of Permutations	Number of Combinations
Where order of elements matters.	Elements can be in any order.
${}_nP_r = \frac{n!}{(n-r)!}$	${}_nC_r = \frac{n!}{r!(n-r)!}$

For problems 1-4, compute the number of permutations and combinations, using a calculator when necessary.

1. ${}_5P_3 = 60$

2. ${}_{10}C_2 = 45$

3. ${}_{20}C_5 = 15,504$

4. ${}_{40}P_{35} \approx 6.80 \times 10^{45}$

5. A student council has 5 seniors, 4 juniors, 2 sophomores, and 2 freshmen as members. In how many ways can a 4-member council committee be formed that includes one member of each class?

$5 \cdot 4 \cdot 2 \cdot 2 = 80$ possible committees

6. How many passwords of five symbols can be made if each password must start with two digits and end three letters given that...

- a. letters and digits may be repeated?

$10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 1,757,600$ passwords

- b. neither the letters nor the digits may be repeated?

$10 \cdot 9 \cdot 26 \cdot 25 \cdot 24 = 1,404,000$ passwords

7. Carla wants to take 6 different courses next year. There are 12 courses to choose from and assuming that they are all offered at different times, how many different course loads could she have?

$${}_{12}C_6 = 924 \text{ course loads}$$

8. A baseball team has 15 players. How many 9-player batting orders are possible?

$${}_{15}P_9 = 1,816,214,400 \text{ batting orders}$$

9. A student activity club has 32 members. In how many different ways can the club select a 5-person committee?

$${}_{32}C_5 = 201,376 \text{ committees}$$

10. A baseball team has 15 players. How many 3-player groups can be chosen to run laps around the field?

$${}_{15}C_3 = 455 \text{ groups}$$

11. A student activity club has 32 members. In how many different ways can the club elect a president, a vice president, a treasurer, and a secretary?

$${}_{32}P_4 = 863,040 \text{ different officer groups}$$

12. A suitcase contains 6 distinct pairs of socks and 4 distinct pairs of pants. In how many ways can a traveler randomly pick 2 pairs of socks and then 3 pairs of pants?

$${}_6C_2 \cdot {}_4C_3 = 15 \cdot 4 = 60 \text{ pants and jackets}$$

Name: _____ Date: _____

Lesson 6: Permutations and Combinations, continued

Assignment (*Student Document*)

For questions 1-6, simplify expressions. Show all work. You may use a calculator to estimate final answers for which the numerator or denominator is larger than 1000.

1. ${}_{16}P_2$

2. ${}_{16}C_2$

3. ${}_{11}C_0$

4. ${}_{10}P_{10}$

5. ${}_{51}C_6$

6. ${}_{46}P_{10}$

For Questions 7-10, state whether each scenario describes a permutation or combination.

7. A team of 8 basketball players needs to choose a captain and co-captain.
8. The batting order for seven players on a 12-person softball team.
9. There are 45 applicants for three Computer Programmer positions.

10. Rob and Mary are planning trips to nine countries this year out of a possible 13 countries. They are deciding which countries to skip.

For Questions 11-15, compute the number of outcomes.

11. Castel and Joe are planning trips to three countries this year. There are 7 countries they would like to visit. One trip will be one week long, another two days, and the other two weeks.
12. A group of 25 people are going to run a race. The top 8 finishers, regardless of their ranking within the top 8, advance to the finals.
13. A team of 24 softball players needs to choose three players to refill the water cooler.
14. Determining the batting order for 11 players on an 11-person cricket team.
15. The student council of 10 students wants to elect a president, vice president, secretary, and treasurer.

Lesson 6: Permutations and Combinations, continued**Assignment (Teacher Document)**

For questions 1-6, simplify the expressions. Show all work. You may use a calculator to estimate final answers for which the numerator or denominator is larger than 1000.

$$1. \quad {}_{16}P_2 = \frac{16!}{14!} = 210$$

$$2. \quad {}_{16}C_2 = \frac{16!}{2!14!} = 120$$

$$3. \quad {}_{11}C_0 = \frac{11!}{0!11!} = 1$$

$$4. \quad {}_{10}P_{10} = \frac{10!}{0!} = 3,628,800$$

$$5. \quad {}_{51}C_6 = \frac{51!}{6!45!} = 18,009,460$$

$$6. \quad {}_{46}P_{10} = \frac{46!}{36!} \approx 1.48 \times 10^{16}$$

For Questions 7-10, state whether each scenario describes a permutation or combination.

7. A team of 8 basketball players needs to choose a captain and co-captain.

Permutation

8. The batting order for seven players on a 12-person softball team.

Permutation

9. There are 45 applicants for three Computer Programmer positions.

Combination

10. Rob and Mary are planning trips to nine countries this year out of a possible 13 countries. They are deciding which countries to skip.

Combination

For Questions 11-15, compute the number of outcomes.

11. Castel and Joe are planning trips to three countries this year. There are 7 countries they would like to visit. One trip will be one week long, another two days, and the other two weeks.

$${}_7P_3 = 210 \text{ options}$$

12. A group of 25 people are going to run a race. The top 8 finishers, regardless of their ranking within the top 8, advance to the finals.

$${}_{25}C_8 = 1,081,575 \text{ possible top 8 finishing groups}$$

13. A team of 24 softball players needs to choose three players to refill the water cooler.

$${}_{24}C_3 = 2,024 \text{ water groups}$$

14. Determining the batting order for 11 players on an 11-person cricket team.

$${}_{11}P_{11} = 39,916,800 \text{ batting orders}$$

15. The student council of 10 students wants to elect a president, vice president, secretary, and treasurer.

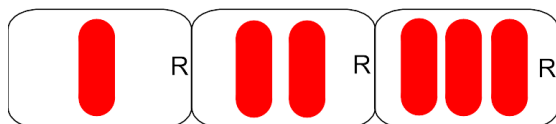
$${}_{10}P_4 = 5,040 \text{ ways}$$

Name: _____ Date: _____

Lesson 7: Counting Challenge

Activity (*Student Document*)

1. How many *SETs* are there in the full *SET*® deck?
 - a. Consider the *SET* below. How many permutations does this *SET* have?



- b. A *SET* is a _____ of cards since the order of the cards does not matter.
 - c. Let's make an arbitrary *SET*! How many options do we have for the first card?
 - d. Once the first card is chosen, are there any restrictions on the second card? How many options are there for the second card?
 - e. Of the _____ cards left after the first two are chosen, how many of those cards will complete a *SET* with the first two cards? (Recall the _____: there is exactly _____ card that will complete a *SET* with any two given cards.)
 - f. There are _____ *ordered* lists of cards that form a *SET*, and thus when counting *SETs* (where order doesn't matter) we must compensate for overcounting by dividing by _____.
 - g. The number of *SETs* in a full *SET*® deck is:

Next, we'll determine how many *SETs* have exactly 1 (respectively 2, 3, or 4) attributes that are different. This is really four questions. First examine the examples of each type of *SET*:

A *SET* with exactly one attribute different (e.g. color):



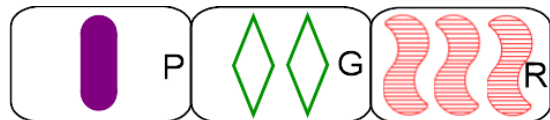
A *SET* with exactly two attributes different (e.g. number and shading):



A *SET* with exactly three attributes different (e.g. number, color, and shading):

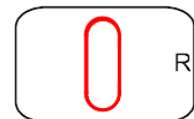


A *SET* with all four attributes different:



2. How many *SETs* have all 4 attributes different?

a. We begin by choosing one card arbitrarily, for example:



b. The second card has to be different in all FOUR features, meaning that it cannot have one symbol, be red, have empty shading or be an oval. These restriction give us 2 choices for each attribute.

c. So there are _____ ways to choose the second card.

- d. Now once again by the _____, after the first two cards are chosen, the third card has already been determined.
- e. The last step is to divide by the number of ways to order a single *SET*, which is _____. Forgetting to do this step would result in an answer larger than the total number of *SETs* in the entire deck (_____).
- f. The number of *SETs* that have all 4 attributes different in a full *SET*® deck is:

3. How many *SETs* have exactly 3 attributes different (one attribute the same)?

- a. We begin by choosing one card arbitrarily, for example:



- b. The second card has to be the same in one feature and different in the three other features.
- c. There are _____ ways to choose which feature is the same, and 2 ways to choose each feature that is different. So by the Fundamental Counting Principle, there are _____ ways to choose the second card.
- d. So there are _____ ways to choose the first two cards. The third card is determined by the first two.
- e. Again, though, we have overcounted different permutations that are the same *SET* so the last step is to divide by the number of ways to order a single *SET*.
- g. Therefore, the number of *SETs* that have exactly 3 attributes that are different in a full *SET*® deck is:

Questions 4 and 5 can be found on the Assignment document.

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 30-31, 34-38.

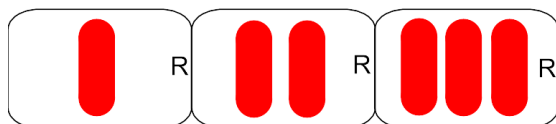
Name: _____ Date: _____

Lesson 7: Counting Challenge

Activity (Teacher Document)

1. How many *SETs* are there in the full SET[®] deck?
- a. Consider the *SET* below. How many permutations does this *SET* have?

6 permutations



- b. A *SET* is a combination of cards since the order of the cards does not matter.
- c. Let's make an arbitrary *SET*! How many options do we have for the first card? 81
- d. Once the first card is chosen, are there any restrictions on the second card? How many options are there for the second card? Any 2nd card will determine a *SET*. So there are 80 choices for the 2nd card.
- e. Of the 79 cards left after the first two are chosen, how many of those cards will complete a *SET* with the first two cards? (Recall the Fundamental Theorem of SET[®]: there is exactly one card that will complete a *SET* with any two given cards.)
- f. There are 81 · 80 · 1 *ordered* lists of cards that form a *SET*, and thus when counting *SETs* (where order doesn't matter) we must compensate for overcounting by dividing by 3! = 3 · 2 · 1 = 6.
- g. The number of *SETs* in a full SET[®] deck is:

$$\frac{81 \cdot 80 \cdot 1}{3!} = \frac{6480}{6} = 1,080$$

Next, we'll determine how many *SETs* have exactly 1 (respectively 2, 3, or 4) attributes that are different. This is really four questions. First examine the examples of each type of *SET*:

A *SET* with exactly one attribute different (e.g. color):



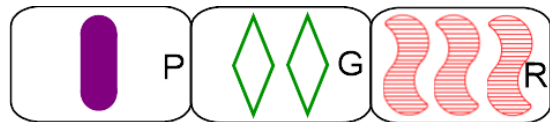
A *SET* with exactly two attributes different (e.g. number and shading):



A *SET* with exactly three attributes different (e.g. number, color, and shading):

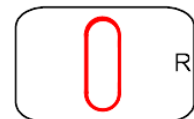


A *SET* with all four attributes different:



2. How many *SETs* have all 4 attributes different?

a. We begin by choosing one card arbitrarily, for example:



b. The second card has to be different in all FOUR features, meaning that it cannot have one symbol, be red, have empty shading or be an oval. These restriction give us 2 choices for each attribute.

c. So there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways to choose the second card.

- d. Now once again by the Fundamental Theorem of SET[®], after the first two cards are chosen, the third card has already been determined.
- e. The last step is to divide by the number of ways to order a single *SET*, which is $3! = 6$. Forgetting to do this step would result in an answer larger than the total number of *SETs* in the entire deck (1080).
- f. The number of *SETs* that have all 4 attributes different in a full SET[®] deck is:

$$\frac{81 \cdot (2 \cdot 2 \cdot 2 \cdot 2) \cdot 1}{3!} = \frac{81 \cdot 16}{6} = 216$$

3. How many *SETs* have exactly 3 attributes different (one attribute the same)?

- a. We begin by choosing one card arbitrarily, for example:



- b. The second card has to be the same in one feature and different in the three other features.
- c. There are ${}_4C_1$ ways to choose which feature is the same, and 2 ways to choose each feature that is different. So by the Fundamental Counting Principle, there are ${}_4C_1 \cdot 2 \cdot 2 \cdot 2$ ways to choose the second card.
- d. So there are $81 \cdot ({}_4C_1 \cdot 2 \cdot 2 \cdot 2)$ ways to choose the first two cards. The third card is determined by the first two.
- e. Again, though, we have overcounted different permutations that are the same *SET* so the last step is to divide by the number of ways to order a single *SET*.
- g. Therefore, the number of *SETs* that have exactly 3 attributes that are different in a full SET[®] deck is:

$$\frac{81 \cdot ({}_4C_1 \cdot 2 \cdot 2 \cdot 2) \cdot 1}{3!} = \frac{81 \cdot (4 \cdot 8)}{6} = 432$$

Questions 4 and 5 can be found on the Assignment document.

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 30-31, 34-38.

Name: _____ Date: _____

Lesson 7: Counting Challenge

Assignment (*Student Document*)

Answer the two remaining questions based on the work from the activity, then fill in the table at the end of this document.

4. How many *SETs* have exactly 2 attributes different (and 2 attributes the same)?
5. How many *SETs* have exactly 1 attribute different (and 3 attributes the same)?

Kind of <i>SET</i>	Frequency	Relative Frequency	Percentage
All different			
Three Different			
Two Different			
One Different			
Totals:	1080	1	100%

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 30-31, 34-38.

Lesson 7: Counting Challenge**Assignment (Teacher Document)**

Answer the two remaining questions based on the work from the activity, then fill in the table at the end of this document.

4. How many *SETs* have exactly 2 attributes different (and 2 attributes the same)?

As always, the first card has no limitations.

There are ${}_4C_2$ ways to choose which attributes are the same. There are 2 choices for each of the attributes that are different.

So there are $({}_4C_2)(2 \cdot 2)$ total ways to choose the second card so that two attributes remain the same and two differ.

The third card once again is already determined and we must divide by $3!$ to avoid overcounting.

Thus, there are $\frac{81 \cdot ({}_4C_2)(2 \cdot 2) \cdot 1}{3!} = \frac{81 \cdot (6 \cdot 4)}{6} = 324$ *SETs* with two attributes different.

5. How many *SETs* have exactly 1 attribute different (and 3 attributes the same)?

As always, the first card has no limitations.

There are ${}_4C_3$ ways to choose which attributes remain the same. For the attribute that changes, there are two options that are different from the first card.

So there are $({}_4C_3)(2)$ total ways to choose the second card so that 3 attributes remain the same and 1 differs.

The third card once again is already determined and we must divide by $3!$ to avoid overcounting.

Thus, there are $\frac{81 \cdot ({}_4C_3)(2) \cdot 1}{3!} = \frac{81 \cdot (4 \cdot 2)}{6} = 108$ *SETs* with one attribute different.

Kind of <i>SET</i>	Frequency	Relative Frequency	Percentage
All different	216	.2	20%
Three Different	432	.4	40%
Two Different	324	.3	10%
One Different	108	.1	10%
Totals:	1080	1	100%

Reference for this document:

McMahon, Liz, Gary Gordon, Hannah Gordon, and Rebecca Gordon. *The Joy of SET®: The Many Mathematical Dimensions of a Seemingly Simple Card Game*. Princeton: Princeton University Press, 2017. Print. Pages 30-31, 34-38.

Appendix B: Cumulative Study Guide

Name: _____ Date: _____

Cumulative Study Guide

(Student Document)

Vocabulary:

Use the space below to record any definitions, notes and examples of the following terms.

Fundamental Theorem of SET[®]:

Example:

Fundamental Counting Principle:

Example:

The factorial function:

Example:

Permutation:

Example:

Combination:

Example:

Computations:

Simplify the following expressions. Show all work. You may use a calculator to estimate final answers for which the numerator or denominator is larger than 1000.

a. $6!$

b. $4! \cdot 5!$

c. $\frac{2! \cdot 8!}{0!}$

d. $20!$

e. $\frac{3!}{6!}$

f. $(3!)!$

g. $0!$

h. $\frac{12!}{10!}$

i. $\frac{95!}{97!}$

j. $6! + 3!$

k. $\frac{14!}{15!}$

l. $\frac{(4!)^2}{16!}$

m. $6! - 5!$

n. $(2!)^2$

o. $\frac{34! \cdot 4!}{16!}$

p. ${}_6P_2$

q. ${}_{12}P_8$

r. ${}_{10}P_{10}$

s. ${}_{10}P_0$

t. ${}_9C_5$

u. ${}_{20}C_{15}$

v. ${}_{12}C_0$

Combinatorics:

For the following examples, compute the number of outcomes.

- a. There are eight seniors on the football team who are being considered as team captains. If there will be 3 team captains, how many different ways can 3 of these seniors be chosen as captains?
- b. There are 5 people on a bowling team. How many different ways are there to arrange the order in which they bowl?
- c. There are 5 people on your bowling team. In how many ways can you choose a bowling team captain and team manager?
- d. You are eating dinner at a restaurant. The restaurant offers 6 appetizers, 12 main dishes, 6 side orders, and 8 desserts. If you order one of each of these, how many different dinners can you order?
- e. A pizza parlor has a special on a three-topping pizza. If you can only have one of each topping and there are 12 toppings total, how many different three-topping pizzas can you order?
- f. Find the number of committees of 4 people that a group of 30 students could have.
- g. How many different seven-digit telephone numbers can be formed if the first digit cannot be 0 or 1?

- h. How many different five-digit ZIP codes are there if any digit can be used in any position?
- i. How many ways can you arrange the letters in the word REPEAT to create six-letter “words”?
- j. How many ways can you arrange the letters in the word THURSDAY to create eight-letter “words”?
- k. An amusement park has 20 different roller coasters. How many different groups of 15 roller coasters can you choose to ride?
- l. A traveler has 18 pairs of socks and 4 pairs of shoes, but he can only fit 5 pairs of socks and 2 pairs of shoes in his suitcase. How many ways can he pack socks and shoes?
- m. At a local pizza shop, they have a special deal for each day of the week. Today’s deal is “Two-Topping Tuesday”, where you have to get exactly 2 toppings from each category: cheese, meat, and vegetables. If there are 3 types of cheese, 6 meats and 11 vegetables, how many unique “Two-Topping Tuesday” pizzas are possible to make?

Counting Challenge:

Use the space below to calculate in clear and concise steps how many SETs there are in a standard deck of SET[®] cards.

Name: _____ Date: _____

Cumulative Study Guide

(Teacher Document)

Vocabulary:

Use the space below to record any definitions, notes and examples of the following terms.

Fundamental Theorem of SET[®]: Given any pair of cards, there is a unique third card that completes a *SET* with the given pair.

Example:

Fundamental Counting Principle: If one events can occur m ways and another event can occur n ways, then the two events together can occur $m \cdot n$ ways.

Example:

The factorial function: For any integer $n \geq 0$, $n!$ is the product of all the positive integers less than or equal to n . By definition, $0! = 1$.

Example:

Permutation: An ordered arrangement (or list) of elements taken from a set.

Example:

Combination: An unordered collection of elements taken from a set.

Example:

Computations:

Simplify the following expressions. Show all work. You may use a calculator to estimate final answers for which the numerator or denominator is larger than 1000.

a. $6! = 720$

b. $4! \cdot 5! = 2,880$

c. $\frac{2! \cdot 8!}{0!} = 80,640$

d. $20! \approx 2.43 \times 10^{18}$

e. $\frac{3!}{6!} = \frac{1}{120}$

f. $(3!)! = 720$

g. $0! = 1$

h. $\frac{12!}{10!} = 132$

i. $\frac{95!}{97!} = \frac{1}{9,312}$

j. $6! + 3! = 726$

k. $\frac{14!}{15!} = \frac{1}{15}$

$$\text{l. } \frac{(4!)^2}{16!} \approx 2.75 \times 10^{-11}$$

$$\text{m. } 6! - 5! = 600$$

$$\text{n. } (2!)^2 = 4$$

$$\text{o. } \frac{34! \cdot 4!}{16!} = \frac{2}{105}$$

$$\text{p. } {}_6P_2 = 30$$

$$\text{q. } {}_{12}P_8 = 19,958,400$$

$$\text{r. } {}_{10}P_{10} = 3,628,800$$

$$\text{s. } {}_{10}P_0 = 1$$

$$\text{t. } {}_9C_5 = 126$$

$$\text{u. } {}_{20}C_{15} = 15,501$$

$$\text{v. } {}_{12}C_0 = 1$$

Combinatorics:

For the following examples, compute the number of outcomes.

- a. There are eight seniors on the football team who are being considered as team captains. If there will be 3 team captains, how many different ways can 3 of these seniors be chosen as captains?

$${}_8C_3 = 56 \text{ ways to choose captains}$$

- b. There are 5 people on a bowling team. How many different ways are there to arrange the order in which they bowl?

$${}_5P_5 = 120 \text{ different bowling orders}$$

- c. There are 5 people on your bowling team. In how many ways can you choose a bowling team captain and team manager?

$${}_5P_2 = 20 \text{ different ways to choose the captain and team manager}$$

- d. You are eating dinner at a restaurant. The restaurant offers 6 appetizers, 12 main dishes, 6 side orders, and 8 desserts. If you order one of each of these, how many different dinners can you order?

$$6 \cdot 12 \cdot 6 \cdot 8 = 3,456 \text{ different meals}$$

- e. A pizza parlor has a special on a three-topping pizza. If you can only have one of each topping and there are 12 toppings total, how many different three-topping pizzas can you order?

$${}_{12}C_3 = 220 \text{ different pizzas}$$

- f. Find the number of committees of 4 people that a group of 30 students could have.

$${}_{30}C_4 = 27,405 \text{ possible committees}$$

- g. How many different seven-digit telephone numbers can be formed if the first digit cannot be 0 or 1?

$$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8,000,000 \text{ different telephone numbers}$$

- h. How many different five-digit ZIP codes are there if any digit can be used in any position?

$$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100,000 \text{ different ZIP codes}$$

- i. How many ways can you arrange the letters in the word REPEAT to create six-letter “words”?

$$\frac{6!}{2!} = 360 \text{ “words”}$$

- j. How many ways can you arrange the letters in the word THURSDAY to create eight-letter “words”?

$$\frac{8!}{0!} = 40,320 \text{ “words”}$$

- k. An amusement park has 20 different roller coasters. How many different groups of 15 roller coasters can you choose to ride?

$${}_{20}C_{15} = 15,504 \text{ possible ride combinations}$$

- l. A traveler has 18 pairs of socks and 4 pairs of shoes, but he can only fit 5 pairs of socks and 2 pairs of shoes in his suitcase. How many ways can he pack socks and shoes?

$$({}_{18}C_5)({}_4C_2) = 8568 \cdot 6 = 51,408 \text{ socks and shoes combinations}$$

- m. At a local pizza shop, they have a special deal for each day of the week. Today’s deal is “Two-Topping Tuesday”, where you have to get exactly 2 toppings from each category: cheese, meat, and vegetables. If there are 3 types of cheese, 6 meats and 11 vegetables, how many unique “Two-Topping Tuesday” pizzas are possible to make?

$$({}_3C_2) \cdot ({}_6C_2) \cdot ({}_{11}C_2) = 3 \cdot 15 \cdot 55 = 2475 \text{ pizzas}$$

Counting Challenge:

Use the space below to calculate in clear and concise steps how many SETs there are in a standard deck of SET® cards.

SETs are made up of three specialized cards, so consider the number of options for each card.

The first card could be anything out of the deck, so there are 81 options.

The second card can be anything that is left, so 80 options.

The third card is already determined because of the Fundamental Theorem of SET®, so there is only one option for this third card.

This calculation is for the number of ordered SETs; since SETs are unordered though, we must divide by the number of ways to order three cards (3!).

Thus, there are $\frac{81 \cdot 80 \cdot 1}{3!} = \frac{81 \cdot 80 \cdot 1}{3 \cdot 2 \cdot 1} = 1,080$ SETs in a full deck.

**For another approach see the Lesson 7 teacher notes. **

Appendix C: Summative Exam

Name: _____ Date: _____

Summative Exam

(Student Document)

1. Compare and contrast permutations and combinations. Describe specific examples for each as well as how the numbers of each are calculated:
2. Simplify the following expressions without a calculator:
 - a. $5!$
 - b. $0!$
 - c. $\frac{4!}{6!}$
 - d. $\frac{74!}{72!}$
 - e. ${}_{10}P_2$
 - f. ${}_8C_4$

3. Determine whether each example below describes a permutation or a combination. Circle your answer.

- a. The number of ways you can choose a group of 3 puppies to adopt from the animal shelter when there are 20 different puppies to choose from.

Permutation

Combination

- b. The number of ways you could award 1st, 2nd, and 3rd place medals for the science fair.

Permutation

Combination

- c. The number of seven-digit phone numbers that can be made using the digits 0 through 9.

Permutation

Combination

- d. The number of ways a committee of 3 could be chosen from a group of 20.

Permutation

Combination

- e. The number of ways a president, vice-president, and treasurer could be chosen from a group of 20.

Permutation

Combination

For Questions 4-9, compute the number of outcomes.

4. Little Caesar's is offering a special where you can buy a large pizza with one cheese, and one meat for \$7.99. There are 3 kinds of cheese, 9 vegetables, and 5 meats to choose from. How many different variations of the pizza special are possible?
5. Nine people in your class want to be on a 5-person bowling team to represent the class. How many different teams can be chosen?

6. Determine how many ways the students in a math club can elect a president, vice president, and treasurer if the club has 7 members.
7. There are 11 people on a cricket team. How many ways are there to choose 5 players to clean up after practice?
8. There are 13 people on a softball team. How many ways are there to assign them to a nine person batting order?
9. How many ways can you arrange the letters in the word STATISTICS to create ten-letter "words"?
10. Use the space below to calculate in clear and concise steps how many *SETs* there are in a standard deck of SET[®] cards.

Name: _____ Date: _____

Summative Exam

(Teacher Document)

1. Compare and contrast permutations and combinations. Describe specific examples for each as well as how the numbers of each are calculated:

The same elements listed in different orders yield different permutations but the same combination. When order matters, we're counting permutations. The elements of a combination can be listed in *any order*. The number of permutations of r elements taken from a set of n elements is: ${}_nP_r = \frac{n!}{(n-r)!}$. The number of combinations of r elements taken from a set of n elements is: ${}_nC_r = \frac{n!}{r!(n-r)!}$. The only difference in the calculation is the $r!$ in the denominator of the formula for the number of combinations. There are $r!$ different permutations for each combination of r elements. So the number of permutations is $r!$ times the number of combinations.

2. Simplify the following expressions without a calculator:

a. $5! = 120$

b. $0! = 1$

c. $\frac{4!}{6!} = \frac{1}{6 \cdot 5} = \frac{1}{30}$

d. $\frac{74!}{72!} = 74 \cdot 73 = 5,402$

e. ${}_{10}P_2 = \frac{10!}{8!} = 10 \cdot 9 = 90$

f. ${}_8C_4 = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70$

3. Determine whether each example below describes a permutation or a combination. Circle your answer.

- a. The number of ways you can choose a group of 3 puppies to adopt from the animal shelter when there are 20 different puppies to choose from.

Permutation

Combination

- b. The number of ways you could award 1st, 2nd, and 3rd place medals for the science fair.

Permutation

Combination

- c. The number of seven-digit phone numbers that can be made using the digits 0 through 9.

Permutation

Combination

- d. The number of ways a committee of 3 could be chosen from a group of 20.

Permutation

Combination

- e. The number of ways a president, vice-president, and treasurer could be chosen from a group of 20.

Permutation

Combination

For Questions 4-9, compute the number of outcomes.

4. Little Caesar's is offering a special where you can buy a large pizza with one cheese, one meat, and one vegetable for \$7.99. There are 3 kinds of cheese, 9 vegetables, and 5 meats to choose from. How many different variations of the pizza special are possible?

$$3 \cdot 9 \cdot 5 = 135 \text{ different pizza options}$$

5. Nine people in your class want to be on a 5-person bowling team to represent the class. How many different teams can be chosen?

$${}_9C_5 = 126 \text{ different bowling teams}$$

6. Determine how many ways the students in a math club can elect a president, vice president, and treasurer if the club has 7 members.

$${}_7P_3 = 210 \text{ ways to elect president, vice president and treasurer}$$

7. There are 11 people on a cricket team. How many ways are there to choose 5 players to clean up after practice?

$${}_{11}C_5 = 462 \text{ ways to choose water folks}$$

8. There are 13 people on a softball team. How many ways are there to assign them to a 9-person batting order?

$${}_{13}P_9 = 259,459,200$$

9. How many ways can you arrange the letters in the word STATISTICS to create ten-letter “words”?

$$\frac{10!}{3!3!2!} = 50,400 \text{ “words”}$$

10. Use the space below to calculate in clear and concise steps how many *SETs* there are in a standard deck of SET® cards.

SETs are made up of three specialized cards, so consider the number of options for each card.

The first card could be anything out of the deck, so there are 81 options.

The second card can be anything that is left, so 80 options.

The third card is already determined because of the Fundamental Theorem of SET®, so there is only one option for this third card.

This calculation is for the number of ordered *SETs*; since *SETs* are unordered though, we must divide by the number of ways to order three cards (3!).

$$\text{Thus, there are } \frac{81 \cdot 80 \cdot 1}{3!} = \frac{81 \cdot 80 \cdot 1}{3 \cdot 2 \cdot 1} = 1,080 \text{ SETs in a full deck.}$$

**For another approach see the Lesson 7 teacher notes. **

Appendix D: Reference Sheet (SET® Deck)

Name: _____ Date: _____

Reference Sheet (SET® Deck)

(Student Document)

Keep this reference sheet with you at all times for the duration of this unit!

Answer the following questions about a standard SET® deck of cards:

a. How many cards are in a standard SET® deck? _____

b. List the four attributes of SET® cards and list the possible values of each attribute:

c. How many red cards are there?

d. How many cards have diamonds on them?

e. How many red diamond cards are there?

f. How many red striped diamond cards are there?

g. How many three red striped diamond cards are there?

Use the space below to sketch or write three examples of SETs:

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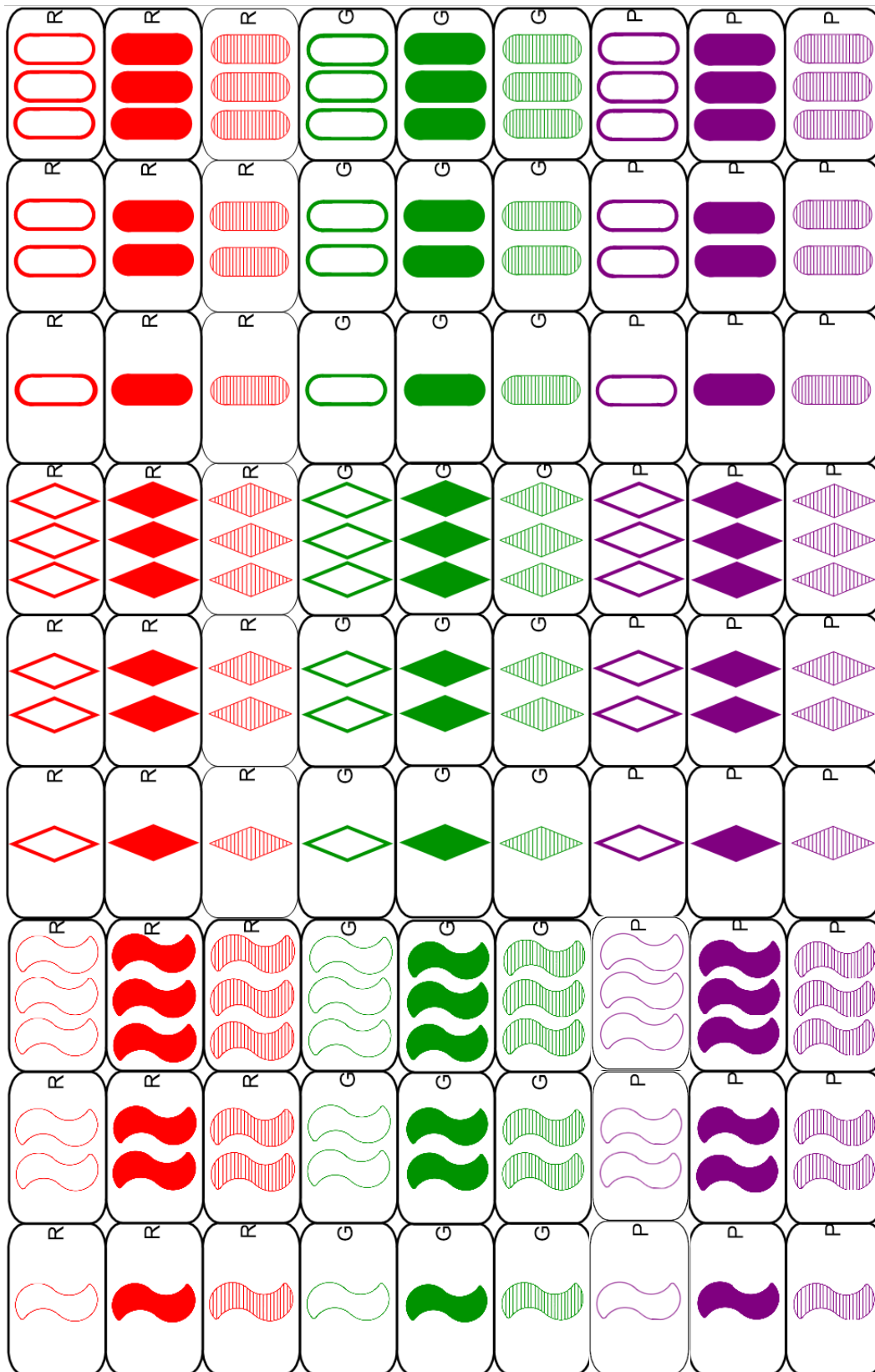
Use the space below to sketch three examples of three cards that do not form *SETs*:

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Complete SET® Deck



Name: _____ Date: _____

Reference Sheet (SET® Deck)

(Student Document)

Keep this reference sheet with you at all times for the duration of this unit!

Answer the following questions about a standard SET® deck of cards:

- a. How many cards are in a standard SET® deck? 81
- b. List the four attributes of SET® cards and list the possible values of each attribute:

number: 1, 2 or 3

shading: solid, empty or striped

color: green, purple or red

shape: oval, squiggle or diamond

- c. How many red cards are there? 27
- d. How many cards have diamonds on them? 27
- e. How many red diamond cards are there? 9
- f. How many red striped diamond cards are there? 3
- g. How many three red striped diamond cards are there? 1

Use the space below to sketch or write three examples of SETs:

Responses will vary.

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Use the space below to sketch three examples of three cards that do not form *SETs*:

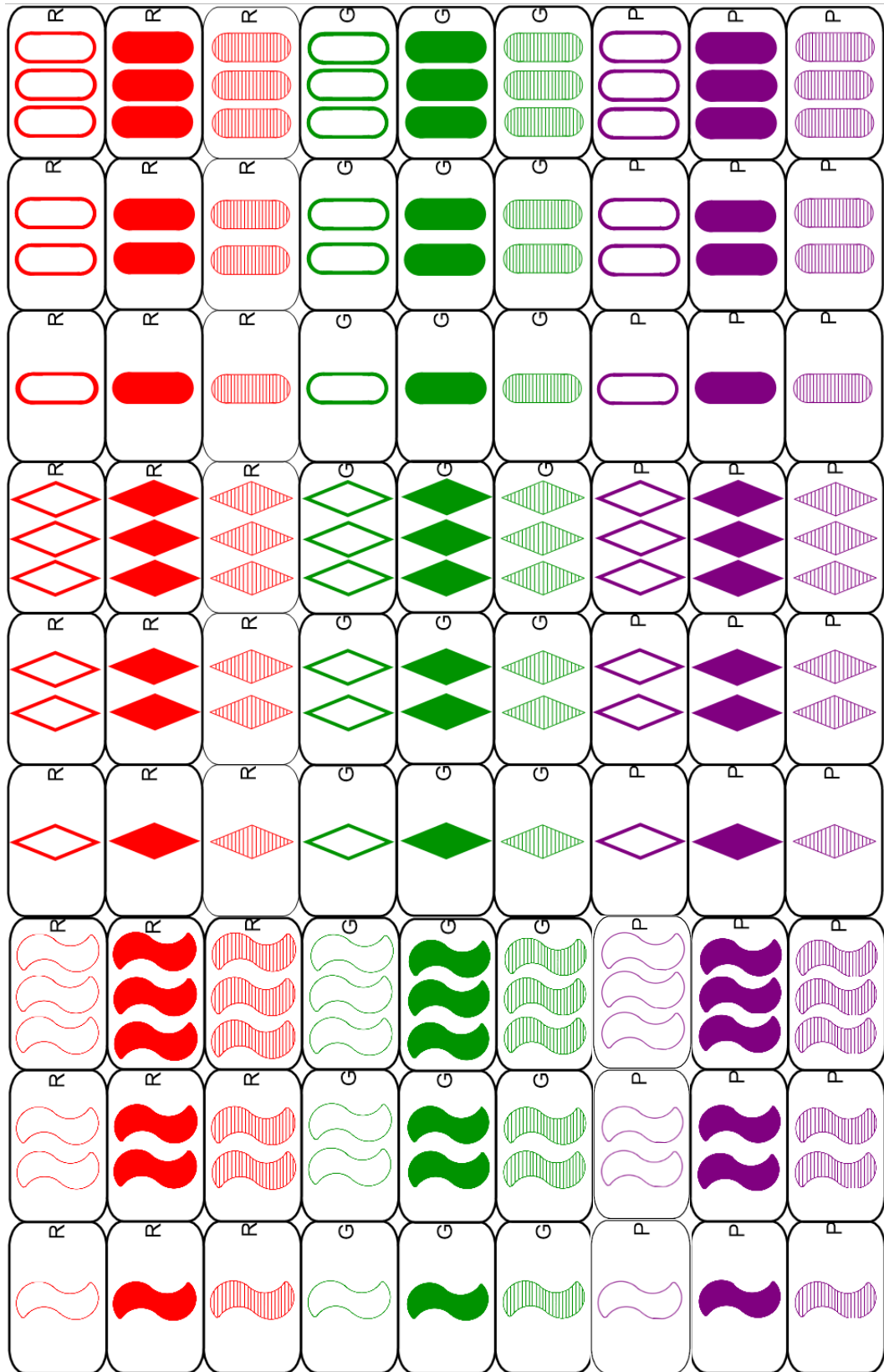
Responses will vary.

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Complete SET® Deck



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